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Non-classical correlation of the two emission fields by a four-level atomic ensemble

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Abstract. The normalized second-order correlation of the emission fields from a driven four-level atomic ensemble is investigated theoretically by using the state vector method. The violation of Cauchy-Schwarz inequality, which indicates the establishment of non-classical correlation between two emission fields, has been found. The effects of various decays and time delay on the correlation are discussed in detail, which are helpful in finding the ways to obtain high non-classical correlation. This technique for the generation of non-classical light is operable based on the current experimental technology and will lead to some potential applications in quantum information science.

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QICS. 18.10.+b System-bath interaction – 33.80.+m Atomic-ensemble quantum memory for light

1 Introduction

Quantum correlation, or entanglement plays a key role in the quantum information science [1]. How to generate quantum-mechanically correlated photon pairs as information carriers is a basic problem in this field. As early as 1970s, Clauser experimentally investigated the non-classically correlated photon pairs in an atomic cascade [2]. Then the technique of four-wave mixing was introduced to produce squeezed states [3–5], which has become one of the most important sources of non-classical lights [6–8]. With the help of measurement technique proposed by Hong, Ou and Mandel [9], degenerate parametric amplification [10–12] is another feasible way to produce non-classical lights [13–15]. Up to present, how to efficiently generate non-classical photon pairs is still a hot research topic [16].

Another difficulty in quantum information science is quantum communication. Due to losses and de-coherence, the entanglement generated between two distant sites decreases exponentially with the length of the photonic channels [17], which makes long-distance quantum communication nearly impossible. To overcome this problem, the concept of quantum repeater is introduced [17]. Duan et al. proposed a probabilistic scheme with built-in quantum memory [18–21] and entanglement purification to generate entanglement over long distances [22]. Recently, some related experimental progresses have been made. Kuzmich and van der Wal et al. reported the generation of correlated photon pairs in the atomic ensemble of cesium and rubidium, respectively [23,24].

In the experiments mentioned above [23,24], the correlated photon pairs are generated via two sequential Raman scatterings in a three-level system. For a four-level atomic system, the performances such as switching and information storage are investigated via optical bistability [25]. It is shown that the shift of optical bistability hysteresis curve can be controlled under the suitable tuning of the controlling lights and the storage of optical signals can be accomplished by alternately triggering the up- and down-controlling optical pulses. In this paper, we focus on the correlation between the two emission fields in a four-level atomic ensemble. The interval between the two external pumping pulses determines the time delay of the two emission fields.

2 The state vector of the photon-atom interaction system

The ensemble under consideration is composed of N ($N = 50\,000$ is set in our numerical calculation) identical atoms. We assume all atoms share the same probability to be excited by the external lasers [26]. The upper levels $|e_1\rangle$ and $|e_2\rangle$ are coupled to two lower levels $|g_1\rangle$ and $|g_2\rangle$ via the vacuum mode k and q, and other transitions are dipole-forbidden. The whole interaction is of four

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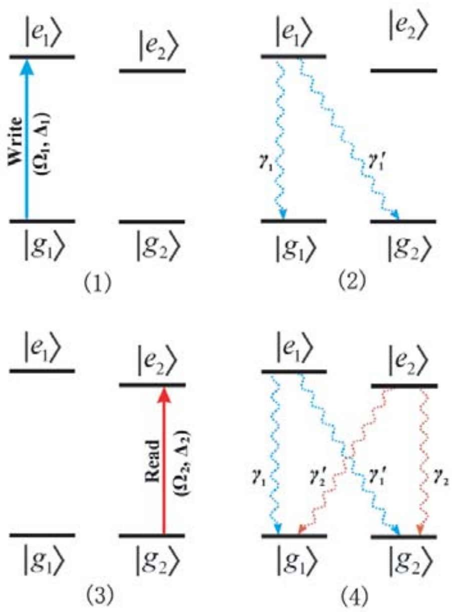

Fig. 1. Level scheme for a driven four-level atomic system. The four decays from two excited states $|e_1\rangle$ and $|e_2\rangle$ to two degenerate ground states $|g_1\rangle$ and $|g_2\rangle$ are dipole-allowed, with transition frequency ω_1 and ω_2.

sequential processes, as shown in Figure 1. Two external laser pulses with Rabi frequency Ω_1, Ω_2 and detuning Δ_1, Δ_2, which we refer to as the *write* and *read* pulses, drive the transitions $|g_1\rangle \rightarrow |e_1\rangle$ and $|g_2\rangle \rightarrow |e_2\rangle$ in the first and third processes. The decays during the two pumping processes [Figs. 1(1) and 1(3)] are omitted, because the durations of the two pumping processes, T_1 and T_2, are assumed very short, and their effects in these two processes will be discussed in Section 4. Processes 2 and 4 [see Figs. 1(2) and 1(4)] are spontaneous decays with durations of T_2 and T_4.

The rotating wave-approximation Hamiltonian of the first process [Fig. 1(1)] can be written in the interaction picture as [27]:

$$V_1 = i\Omega_1 e^{-i\Delta_1 t}\sum_{j=1}^{N} e^{-ik_{11}z_j}|e_1^j\rangle\langle g_1^j| + h.c., \qquad (1)$$

where z_j is the z-component of the jth atom's position $\boldsymbol{r}_j$, and the wave vector of the pumping field is assumed to be along the z-axis. The initial state of the system is supposed to be the tensor product of the ensemble's ground state with all atoms in level $|g_1\rangle$ and the vacuum field,

$$|\phi_1(0)\rangle = |g_1^1 \cdots g_1^N\rangle|0\rangle. \qquad (2)$$

By substituting the formula (1) into Schrödinger equation we can easily get the time dependent state vector in this process bases on the initial condition (2),

$$\begin{aligned}&|\phi_1(t)\rangle = \\ &(A(t))^N|g_1^1\cdots g_1^N\rangle|0\rangle + \sum_{i=1}^{N}(A(t))^{N-1}B(t)|\cdots e_1^i\cdots\rangle|0\rangle \\ &+\sum_{i=1}^{N}\sum_{ii\neq i}^{N}\frac{1}{2}(A(t))^{N-2}B(t)^2|\cdots e_1^i\cdots e_1^{ii}\cdots\rangle|0\rangle+\cdots,\end{aligned} \qquad (3)$$

where $|\cdots e_1^i \cdots e_1^{ii}\cdots\rangle$ stand for only the ith and iith atoms in the exited state (the rest atoms in level $|g_1\rangle$). Functions $A(t)$ and $B(t)$ are $A(t) = \frac{\lambda_1 e^{\lambda_2 t}-\lambda_2 e^{\lambda_1 t}}{\lambda_1-\lambda_2}$ and $B(t) = \frac{\Omega_1 e^{-ik_{11}z_0}(e^{-\lambda_2 t}-e^{-\lambda_1 t})}{\lambda_1-\lambda_2}$, with λ_1 and λ_2 being the two roots of quadratic equation, $\lambda^2 - i\Delta_1\lambda + |\Omega_1|^2 = 0$. Compared with the wave length of the pumping laser, a very small size for the ensemble is assumed for simplicity, which means that the phase factor term $e^{\pm ik_{11}z_j}$ for the jth atom in the ensemble has been considered as a constant $e^{\pm ik_{11}z_0}$ in the calculation. Owing to the short duration T_1, the excitation is very weak, and consequently we have $A(t \leq T_1) \sim 1$ and $B(t \leq T_1) \sim 0$. Therefore, the probability of finding three or more excited atoms in the ensemble is extremely small and can be neglected.

In the second process, the upper level $|e_1\rangle$ is coupled to the two lower levels $|g_1\rangle$ and $|g_2\rangle$ via the vacuum mode k, but with different coupling constant g_k and g'_k [Fig. 1(2)]. For each process we use its own time coordinate, and the "initial" state of the present process is the "final" state of the previous one. Therefore, the initial state of the second process is (T_1 is the duration of the first process),

$$|\phi_2(0)\rangle = |\phi_1(T_1)\rangle. \qquad (4)$$

The rotating wave-approximation Hamiltonian of the second process can be written as:

$$\begin{aligned}V_2 = &\, i\sum_{j=1}^{N}\sum_{k_j} g_{k_j}^{*}e^{i\boldsymbol{k}_j\cdot\boldsymbol{r}_j}e^{i(\omega_1-\omega_{k_j})t}a_{k_j}|e_1^j\rangle\langle g_1^j| \\ &+ i\sum_{j=1}^{N}\sum_{k_j} g_{k_j}^{\prime *}e^{i\boldsymbol{k}_j\cdot\boldsymbol{r}_j}e^{i(\omega_1-\omega_{k_j})t}a_{k_j}|e_1^j\rangle\langle g_2^j| + h.c.,\end{aligned} \qquad (5)$$

where a_{k_j} ($a_{k_j}^{\dagger}$) is the annihilation (creation) operator for the k_jth vacuum mode. Using the Weisskopf-Wigner theory [28], we can get the result of the state vector at time t in the second process under the assumption that all the atoms are confined in a small region,

$$
\begin{aligned}
|\phi_2(t)\rangle = {} & x_0^{(2)}(0)|g_1^1\cdots g_1^N\rangle|0\rangle \\
& + \sum_{i=1}^{N} x_i^{(2)}(0)e^{-\frac{1}{2}(\gamma_1+\gamma_1')t}|\cdots e_1^i\cdots\rangle|0\rangle \\
& + \sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}} 2x_{i,ii}^{(2)}(0) \\
& \times \int_0^t g'_{k_{ii}} e^{-i\boldsymbol{k}_{ii}\cdot\boldsymbol{r}_0} e^{-i(\omega_1-\omega_{k_{ii}})t'} e^{-\frac{1}{2}(\gamma_1+\gamma_1')t'} \\
& \times \int_0^{t'} g_{k_i} e^{-i\boldsymbol{k}_i\cdot\boldsymbol{r}_0} e^{-i(\omega_1-\omega_{k_i})t''} \\
& \times e^{-\frac{1}{2}(\gamma_1+\gamma_1')t''} dt''dt'|\cdots g_2^i\cdots g_1^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle \\
& + \cdots, \qquad (6)
\end{aligned}
$$

where $x_0^{(2)}(0)$, $x_i^{(2)}(0)$ and $x_{i,ii}^{(2)}(0)$ are the initial coefficients of terms $|g_1^1\cdots g_1^N\rangle|0\rangle$, $|\cdots e_1^1\cdots\rangle|0\rangle$ and $|\cdots e_1^i\cdots e_1^{ii}\cdots\rangle|0\rangle$ in this process, which are determined by the second process's initial condition (4), i.e., $x_0^{(2)}(0) = x_0^{(1)}(T_1)$, $x_i^{(2)}(0) = x_i^{(1)}(T_1)$ and $x_{i,ii}^{(2)}(0) = x_{i,ii}^{(1)}(T_1)$. The superscript of the coefficients represents the sequence number of process. $\gamma_1 = \frac{1}{4\pi\varepsilon_0}\frac{4\omega_1^3\wp_{e_1g_1}}{3\hbar c^3}$ and $\gamma_1' = \frac{1}{4\pi\varepsilon_0}\frac{4\omega_1^3\wp_{e_1g_2}}{3\hbar c^3}$ are the decay rates from $|e_1\rangle$ to $|g_1\rangle$ and $|g_2\rangle$, respectively. We have written $\sum_{k_i}|2_{k_i}\rangle$ as $\sum_{k_i}\sum_{k_{ii}=k_i}|1_{k_i}1_{k_{ii}}\rangle$ in the above expression.

The interaction in the third process is similar to the one described in equation (1) [Fig. 1(3)]. Another laser pulse (*read* pulse) with Rabi frequency Ω_2 and detuning Δ_2 pumps level $|g_2\rangle$ to level $|e_2\rangle$. Following a similar calculation, we can derive the time dependent state vector in this process (please see Appendix C and Eq. (C.1)),

$$
\begin{aligned}
|\phi_3(t)\rangle = {} & x_0^{(3)}(0)|g_1^1\cdots g_1^N\rangle|0\rangle + \sum_{i=1}^{N} x_i^{(3)}(0)|\cdots e_1^i\cdots\rangle|0\rangle \\
& + \sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}} Z_{2k_ik_{ii}}^{(3)}(0)(C(t))^2|\cdots g_2^i\cdots g_2^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle \\
& + \sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}} Z_{2k_ik_{ii}}^{(3)}(0) \\
& \times (D(t))^2|\cdots e_2^i\cdots e_2^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle + \cdots, \qquad (7)
\end{aligned}
$$

where functions $C(t)$ and $D(t)$ are defined by $C(t) = \frac{\lambda_3 e^{\lambda_4 t}-\lambda_4 e^{\lambda_3 t}}{\lambda_3-\lambda_4}$ and $D(t) = \frac{\Omega_2 e^{-ik_{22}z_0}(e^{-\lambda_4 t}-e^{-\lambda_3 t})}{\lambda_3-\lambda_4}$, with λ_3 and λ_4 being the two roots of the quadratic equation $\lambda^2 - i\Delta_2\lambda + |\Omega_2|^2 = 0$. The initial coefficients, $x_0^{(3)}(0)$, $x_i^{(3)}(0)$ and $Z_{2k_ik_{ii}}^{(3)}(0)$ are determined by the final state of the second process, $|\phi_3(0)\rangle = |\phi_2(T_2)\rangle$ (note the duration of the second process T_2).

In the last process, there are four decay channels [Fig. 1(4)]. Except for the vacuum mode k related to the two decays from upper level $|e_1\rangle$, new vacuum modes q couple the transitions from the exited states $|e_2\rangle$ to the two lower levels $|g_1\rangle$ and $|g_2\rangle$. The rotating wave-approximation Hamiltonian of this process is,

$$
\begin{aligned}
V_4 = {} & i\sum_{j=1}^{N}\sum_{k_j} g_{k_j}^* e^{i\boldsymbol{k}_j\cdot\boldsymbol{r}_j} e^{i(\omega_1-\omega_{k_j})t} a_{k_j}|e_1^j\rangle\langle g_1^j| \\
& + i\sum_{j=1}^{N}\sum_{k_j} g_{k_j}'^* e^{i\boldsymbol{k}_j\cdot\boldsymbol{r}_j} e^{i(\omega_1-\omega_{k_j})t} a_{k_j}|e_1^j\rangle\langle g_2^j| \\
& + i\sum_{j=1}^{N}\sum_{q_j} g_{q_j}^* e^{i\boldsymbol{q}_j\cdot\boldsymbol{r}_j} e^{i(\omega_2-\omega_{q_j})t} a_{q_j}|e_2^j\rangle\langle g_2^j| \\
& + i\sum_{j=1}^{N}\sum_{q_j} g_{q_j}'^* e^{i\boldsymbol{q}_j\cdot\boldsymbol{r}_j} e^{i(\omega_2-\omega_{q_j})t} a_{q_j}|e_2^j\rangle\langle g_1^j| + h.c. \qquad (8)
\end{aligned}
$$

g_{q_j} (g'_{q_j}) is the coupling constant between the q_jth vacuum mode with frequency ω_{q_j} and the atomic transition from $|e_2\rangle$ to $|g_2\rangle$ ($|g_1\rangle$); a_{q_j} ($a_{q_j}^\dagger$) is the annihilation (creation) operator for the q_jth vacuum mode. The initial state of this process is $|\phi_4(0)\rangle = |\phi_3(T_3)\rangle$. The state vector in this process is calculated in Appendix A, which is

$$
\begin{aligned}
|\phi_4(t)\rangle = {} & x_0^{(4)}(0)|g_1^1\cdots g_1^N\rangle|0\rangle \\
& + \sum_{i=1}^{N} x_i^{(4)}(0)e^{-\frac{1}{2}(\gamma_1+\gamma_1')t}|\cdots e_1^i\cdots\rangle|0\rangle \\
& + \sum_{i=1}^{N}\sum_{k_i}[y_{2k_i}^{(4)}(0) - x_i^{(4)}(0)\int_0^t g'_{k_i} e^{-i\boldsymbol{k}_i\cdot\boldsymbol{r}_0} e^{-i(\omega_1-\omega_{k_i})t'} \\
& \times e^{-\frac{1}{2}(\gamma_1+\gamma_1')t'}dt']|\cdots g_2^i\cdots\rangle|1_{k_i}\rangle + \cdots, \qquad (9)
\end{aligned}
$$

where $\gamma_2 = \frac{1}{4\pi\varepsilon_0}\frac{4\omega_2^3\wp_{e_2g_2}}{3\hbar c^3}$ and $\gamma_2' = \frac{1}{4\pi\varepsilon_0}\frac{4\omega_2^3\wp_{e_2g_1}}{3\hbar c^3}$ are the decay rates from $|e_2\rangle$ to $|g_2\rangle$ and $|g_1\rangle$. Now we have the time-dependent state vector in all processes and are ready to calculate the observable quantities.

3 The normalized second-order correlation

The normalized second-order correlation (NSOC) function provides an important approach to study the properties of correlated photons or fields in quantum and classical optics, which is defined as [29]:

$$
g^{(2)}(t_1,t_2) = \frac{\langle E^{(-)}(\boldsymbol{r},t_1)E^{(-)}(\boldsymbol{r},t_2)E^{(+)}(\boldsymbol{r},t_2)E^{(+)}(\boldsymbol{r},t_1)\rangle}{\langle E^{(-)}(\boldsymbol{r},t_1)E^{(+)}(\boldsymbol{r},t_1)\rangle\langle E^{(-)}(\boldsymbol{r},t_2)E^{(+)}(\boldsymbol{r},t_2)\rangle}, \qquad (10)
$$

and in our system,

$$
\begin{aligned}
\boldsymbol{E}^{(-)}(\boldsymbol{r},t) = {} & \sum_{i=1}^{N}\sum_{k_i} \hat{\varepsilon}_{k_i}\xi_{k_i} a_{k_i}^\dagger e^{i\omega_{k_i}t - i\boldsymbol{k}_i\cdot(\boldsymbol{r}-\boldsymbol{r}_0)} \\
& + \sum_{i=1}^{N}\sum_{q_i} \hat{\varepsilon}_{q_i}\xi_{q_i} a_{q_i}^\dagger e^{i\omega_{q_i}t - i\boldsymbol{q}_i\cdot(\boldsymbol{r}-\boldsymbol{r}_0)}. \qquad (11)
\end{aligned}
$$

$$g^{(2)}_{1,2}(t,t) = \frac{p_1}{p_2 p_3}, \tag{14a}$$

$$p_1 = \sum_{i=1}^{N}\sum_{k_i}\sum_{q_i}(|R_{1k_iq_i}(\infty)|^2 + |R_{2k_iq_i}(\infty)|^2) + 2\sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}}\sum_{q_i}(2|T_{5k_ik_{ii}q_i}(\infty)|^2 + |T_{6k_ik_{ii}q_i}(\infty)|^2) + 2|T_{7k_ik_{ii}q_i}(\infty)|^2)$$
$$+ 4\sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}}\sum_{q_i}\sum_{q_{ii}}(2|T_{11k_ik_{ii}q_iq_{ii}}(\infty)|^2 + 2|T_{12k_ik_{ii}q_iq_{ii}}(\infty)|^2 + |T_{13k_ik_{ii}q_iq_{ii}}(\infty)|^2), \tag{14b}$$

$$p_2 = \sum_{i=1}^{N}\sum_{k_i}(|y_{1k_i}(\infty)|^2 + |y_{2k_i}(\infty)|^2) + \sum_{i=1}^{N}\sum_{k_i}\sum_{q_i}(|R_{1k_iq_i}(\infty)|^2 + |R_{2k_iq_i}(\infty)|^2)$$
$$+ 2\sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}}\left(2|Z_{1k_ik_{ii}}(\infty)|^2 + 2|Z_{2k_ik_{ii}}(\infty)|^2 + |Z_{3k_ik_{ii}}(\infty)|^2\right)$$
$$+ 2\sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}}\sum_{q_i}(2|T_{5k_ik_{ii}q_i}(\infty)|^2 + |T_{6k_ik_{ii}q_i}(\infty)|^2 + 2|T_{7k_ik_{ii}q_i}(\infty)|^2)$$
$$+ 2\sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}}\sum_{q_i}\sum_{q_{ii}}(2|T_{11k_ik_{ii}q_iq_{ii}}(\infty)|^2 + 2|T_{12k_ik_{ii}q_iq_{ii}}(\infty)|^2 + |T_{13k_ik_{ii}q_iq_{ii}}(\infty)|^2), \tag{14c}$$

$$p_3 = \sum_{i=1}^{N}\sum_{k_i}\sum_{q_i}(|R_{1k_iq_i}(\infty)|^2 + |R_{2k_iq_i}(\infty)|^2) + \sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}}\sum_{q_i}(2|T_{5k_ik_{ii}q_i}(\infty)|^2 + |T_{6k_ik_{ii}q_i}(\infty)|^2 + 2|T_{7k_ik_{ii}q_i}(\infty)|^2)$$
$$+ 2\sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}}\sum_{q_i}\sum_{q_{ii}}(2|T_{11k_ik_{ii}q_iq_{ii}}(\infty)|^2 + 2|T_{12k_ik_{ii}q_iq_{ii}}(\infty)|^2 + |T_{13k_ik_{ii}q_iq_{ii}}(\infty)|^2). \tag{14d}$$

Here the time delay T_2 between the emission of the k-field and q-field has been included.

In this section, we study the intensity correlation between the k-field and q-field at the time when the ensemble goes back to its ground state, corresponding to the situation of taking the limit $t \to \infty$ in equation (9) (please see Appendix C and Eq. (C.2)).

$$|\phi(\infty)\rangle = x_0(\infty)|g_1^1 \cdots g_1^N\rangle|0\rangle + \cdots + \sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}}\sum_{q_i}\sum_{q_{ii}} T_{13k_ik_{ii}q_iq_{ii}}(\infty) \times |\cdots g_1^i \cdots g_2^{ii} \cdots\rangle|1_{k_i}1_{k_{ii}}\rangle|1_{q_i}1_{q_{ii}}\rangle. \tag{12}$$

The density matrix of the emission fields can be obtained by tracing the global density matrix over the atom subsystem,

$$\rho_f = \mathrm{Tr_a}(|\phi(\infty)\rangle\langle\phi(\infty)|) = \langle g_1^1 \cdots g_1^N|\phi(\infty)\rangle\langle\phi(\infty)|g_1^1 \cdots g_1^N\rangle + \sum_{i=1}^{N}\langle\cdots g_2^i \cdots|\phi(\infty)\rangle\langle\phi(\infty)|\cdots g_2^i \cdots\rangle + \sum_{i=1}^{N}\sum_{ii=i+1}^{N}\langle\cdots g_2^i \cdots g_2^{ii} \cdots|\phi(\infty)\rangle\langle\phi(\infty)|\cdots g_2^i \cdots g_2^{ii} \cdots\rangle. \tag{13}$$

Applying equation (13) in (10), we can get the result of NSOC as a function of T_1, T_2 and T_3,

see equations (14) above.

Note that the term $x_0(\infty)|g_1^1 \cdots g_1^N\rangle|0\rangle$ in the state vector has no contribution to the correlation though it is the most probable state that the system will stay in.

4 Results and discussions

4.1 Quantum correlation of the two emission fields

The whole process (see Fig. 1) can be viewed as the combination of two 2-channel scatterings with a finite interval. The *write* pulse illuminates the atomic system in the first scattering and generates the *write* signal pulse around frequency ω_1; while the *read* pulse interrogates the atomic system in the second scattering and produces the *read* signal pulse around frequency ω_2. The controllable time separation T_2 determines the delay between the two signal pulses. In Figure 2, we plot the evolution of NSOC with respect to T_2. For simplicity, the resonance condition ($\Delta_1 = \Delta_2 = 0$) is assumed.

As we all known, if only one of the two emission fields (k-field and q-field) is considered (i.e. tracing the density matrix (13) over the degree of the other field), a thermal state will be derived due to the atoms' random distribution [23]. That is, the auto-correlation of k-field and q-field

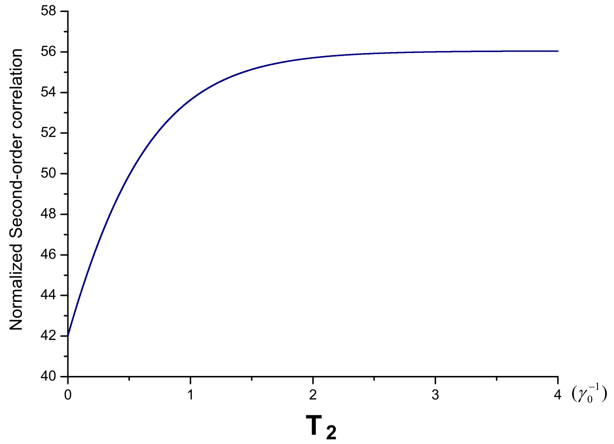

Fig. 2. Results of NSOC versus the time delay of the two signal pulses T_2, for $N = 50\,000$, $\Delta_1 = \Delta_2 = 0$, $\gamma_1 = \gamma_1' = \gamma_2 = \gamma_2' = \gamma_0$, $\Omega_1 = 0.5\gamma_0$, $\Omega_2 = 50\gamma_0$, $T_1 = 0.001/\Omega_1$ and $T_3 = 1.57/\Omega_2$.

is $|g^{(2)}_{1,1}(t,t)|^2 = |g^{(2)}_{2,2}(t,t)|^2 = 4.0$. Classically, the intensity correlation between such two fields should obey,

$$\begin{aligned}\left|g^{(2)}_{1,2}(t,t)\right|^2 &= \left|\frac{\langle I_k(t)I_q(t)\rangle}{\langle I_k(t)\rangle\langle I_q(t)\rangle}\right|^2\\ &\le \frac{\langle |I_k(t)|^2\rangle\langle |I_q(t)|^2\rangle}{|\langle I_k(t)\rangle\langle I_q(t)\rangle|^2}\\ &= g^{(2)}_{1,1}(t,t)g^{(2)}_{2,2}(t,t) = 4.0, \end{aligned} \quad (15)$$

where the symbol $\langle\ \rangle$ donates the operator average. From Figure 2 we see $|g^{(2)}_{1,2}(t,t)|^2 > 4.0$. The value $|g^{(2)}_{1,2}|^2 \equiv |g^{(2)}_{1,2}(t,t)|^2 > 4.0$ for the total field violates the Caucy-Schwarz inequality, i.e., $|g^{(2)}_{1,2}|^2 > g^{(2)}_{1,1}g^{(2)}_{2,2}$, which means that the *write* and *read* signal pulses with a controllable time delay are non-classically correlated [23]. In addition, the long delay (T_2) between the two signal pulses, preserves rather than destroy the quantum correlation.

The coherence between the two lower levels is the key point for the generation of quantum correlation for the emitted fields. At the end of the first scattering (be exited by the *write* pulse in the first process and then spontaneously emit one photon of the first field (k-modes) by decaying to $|g_2\rangle$ in the second process), the atoms stay in the superposition of the two lower levels $|g_1\rangle$ and $|g_2\rangle$ [30]. In the second scattering, the atom emits another photon into the second field (q-modes). The atomic coherence between $|g_1\rangle$ and $|g_2\rangle$ leads to the quantum correlation between the two emission fields.

It is clear that in order to gain strong quantum correlation, we need to have large atomic coherence between $|g_1\rangle$ and $|g_2\rangle$. Therefore, the duration of the second (fourth) process T_2 (T_4) should be much longer than the decay time from $|e_1\rangle$ to $|g_2\rangle$ ($|e_2\rangle$ to $|g_1\rangle$). (Of course, much short than the coherence time between $|g_1\rangle$ and $|g_2\rangle$, which is neglected.) For strong quantum correlation, the *read* pulse is better a π-pulse, which can completely transfer the population in $|g_2\rangle$ to level $|e_2\rangle$, and the decay from $|e_1\rangle$ to $|g_1\rangle$ needs to be small.

Our present calculation will go to the similar results of reference [23] if we simplify our system to the one in reference [23]. First, the two decays $|e_1\rangle \to |g_1\rangle$ and $|e_2\rangle \to |g_2\rangle$ in process 2 and 4 are ignored in reference [23], so we can set $g_{k_i} = g_{q_i} = 0$ and $\gamma_1 = \gamma_2 = 0$ in our calculation; second, $|\Omega_1|T_1 \ll 1$, $T_2 = \infty$ and $|\Omega_2|T_3 \approx \frac{\pi}{2}$ should be satisfied besides the resonant condition ($\Delta_2 = 0$) for the *write* pulse. The difference in level structure between our system and the one in reference [23] have no important influence on the results if the above conditions are met. Substituting these new parameters into the expression (12), we will get the new steady state as,

$$\begin{aligned}|\phi'_{a-f}(\infty)\rangle &= x_0(\infty)|g_1^1\cdots g_1^N\rangle|0\rangle\\ &+\sum_{i=1}^{N}\sum_{k_i}\sum_{q_i}R_{1k_iq_i}(\infty)|\cdots g_1^i\cdots\rangle|1_{k_i}\rangle|1_{q_i}\rangle\\ &+\sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}}\sum_{q_i}\sum_{q_{ii}}T_{12k_ik_{ii}q_iq_{ii}}(\infty)\\ &\times|\cdots g_1^i\cdots g_1^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle|1_{q_i}1_{q_{ii}}\rangle, \end{aligned} \quad (16)$$

and the partial density matrix for the total field becomes,

$$\rho'_f(\infty) = \langle g_1^1\cdots g_1^N|\phi'_{a-f}(\infty)\rangle\langle\phi'_{a-f}(\infty)|g_1^1\cdots g_1^N\rangle. \quad (17)$$

The coefficient $|x_1(\infty)|$ is near unity when $|\Omega_1|T_1 \ll 1$ and $|\Omega_2|T_3 \approx \frac{\pi}{2}$, so the excitation probability is very small, $p = 1 - |x_0(\infty)|^2 \ll 1$. The value of the NSOC in this simplified system is,

$$g'^{(2)}_{1,2}(t,t) \approx \frac{1}{p}. \quad (18)$$

Undoubtedly, the results (16)–(18) are based on the ideal situation. If we take into account some practical factors, such as the decays $|e_1\rangle \to |g_1\rangle$ and $|e_2\rangle \to |g_2\rangle$ in process 2 and 4, the finite time delay T_2, the correlation of the two emission fields will decrease. These explain why strong quantum correlation is not be observed in the experiments. In reference [23], the correlation $|g^{(2)}_{1,2}| \approx 2.335$ is reported, much smaller than the theoretical forecast.

4.2 Effects of the decays

Spontaneous decay is inevitable in quantum optics. In many cases it plays an undesirable role of reducing coherence or destroying entanglement. However, it may also be used to enhance quantum correlations if we have a good understanding about its influence on our system. As indicated in the definition of NSOC (10) and the result (14), the correlation function NSOC is mainly contributed by the coherence terms such as $|1_{k_i}\rangle|1_{q_i}\rangle$, $|1_{k_i}1_{k_{ii}}\rangle|1_{q_i}\rangle$ and $|1_{k_i}1_{k_{ii}}\rangle|1_{q_i}1_{q_{ii}}\rangle$. The single-mode terms, such as $|1_{k_i}\rangle$ and $|1_{k_i}1_{k_{ii}}\rangle$, can be regarded as the uncorrelated parts in the total field. Because all the correlated parts originate from the atomic coherence between $|g_1\rangle$ and $|g_2\rangle$ at the end of the second process, the large decay rate γ_1' of $|e_1\rangle \to |g_2\rangle$

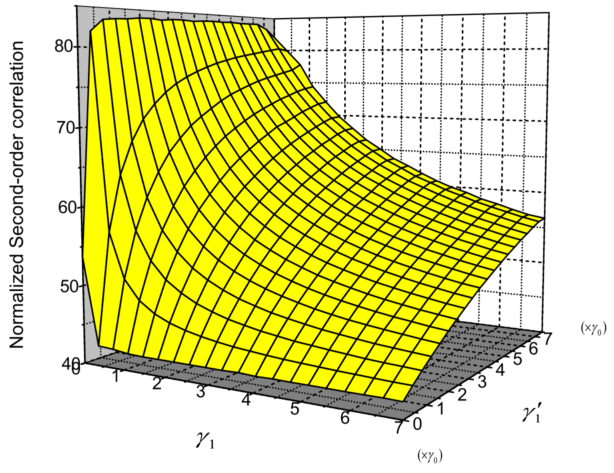

Fig. 3. Results of NSOC versus the decay rates γ_1 and γ_1', for $N = 50\,000$, $\Delta_1 = \Delta_2 = 0$, $\gamma_2 = \gamma_2' = \gamma_0$, $\Omega_1 = 0.5\gamma_0$, $\Omega_2 = 50\gamma_0$, $T_1 = 0.001/\Omega_1$, $T_2 = 20.0/\gamma_1'$ and $T_3 = 1.57/\ \Omega_2$.

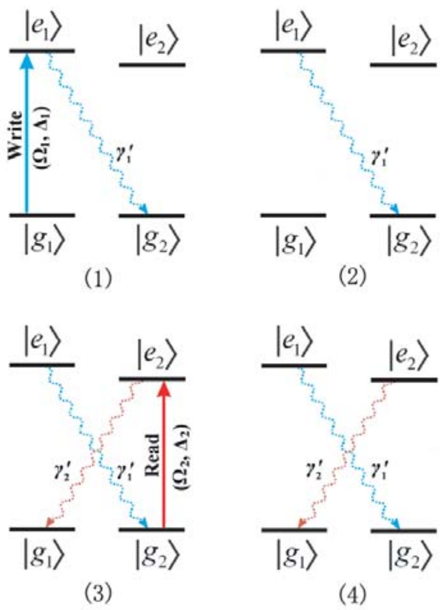

Fig. 4. Level scheme for the new four-level atomic system. Only two transitions $|e_1\rangle \to |g_2\rangle$ and $|e_2\rangle \to |g_1\rangle$ are dipole-allowed, with transition frequency ω_1 and ω_2. Two external laser pulses, with Rabi frequency Ω_1, Ω_2 and detuning Δ_1, Δ_2, drive the transitions $|g_1\rangle \to |e_1\rangle$ and $|g_2\rangle \to |e_2\rangle$, respectively. The *write* and *read* pulses sequentially illuminate the ensemble in the first (1) and the third (3) process, with short duration T_1 and T_3. The second (2) and last (4) processes correspond to the spontaneous decay. The decays during the two pumping processes (1) and (3) are considered.

will leads to large population in $|g_2\rangle$ and large coherence between $|g_1\rangle$ and $|g_2\rangle$, and thus enhance the correlation between the fields. On the contrary, the fast decay $|e_1\rangle \to |g_1\rangle$ reduces the atomic coherence by reducing the population in level $|g_2\rangle$. Hence, the correlation (NSOC) of the two emission fields will enhance with the increasing γ_1' and decreasing γ_1 [see Fig. 3].

Because we only care about the properties of the steady radiation ($|\phi(\infty)\rangle$), each emission from level $|e_2\rangle$ gives birth of one q-mode photon in the last process, no matter which lower level would be chosen to decay to. That is why the correlation of the total steady field will not be affected by the decay rates γ_2 and γ_2'.

Another interesting question is the effects of the decays during the two pumping processes (the first and third processes). As far as we known, there is no analytic solution if we take account of all the four decay channels during the *write* and *read* pumping processes. However, we can study a simpler system to investigate the effects of these decays. The new level scheme is shown in Figure 4. Here we include the decays of $|e_1\rangle \to |g_2\rangle$ and $|e_2\rangle \to |g_1\rangle$ during the pumping processes and neglect the decays of $|e_1\rangle \to |g_1\rangle$ and $|e_2\rangle \to |g_2\rangle$ completely.

The calculation of the state vector of this system is similar to the one made in the previous system. The most complicated interaction occurs in the third process [see Fig. 4(3)]. In Appendix B, we present the detailed calculation in this process. The time-dependent state vector of the last processes in this new system is shown in Appendix C (please refer to Eq. (C.3)). Note that the functions $A(t)$, $B(t)$, $C(t)$ and $D(t)$ in this new system are defined in the same way as the one in the first one. However, λ_1 and λ_2 here are now the two roots of new quadratic equation, $\lambda^2 + (\frac{1}{2}\gamma_1' - i\Delta_1)\lambda + |\Omega_1|^2 = 0$, while λ_3 and λ_4 are the two roots of the quadratic equation, $\lambda^2 + (\frac{1}{2}\gamma_2' - i\Delta_2)\lambda + |\Omega_2|^2 = 0$.

The steady state can be derived by taking the limit $t \to \infty$ in the expression (C.3):

$$\begin{aligned}
|\phi(\infty)\rangle &= x_0(\infty)|g_1^1 \cdots g_1^N\rangle|0\rangle \\
&+ \sum_{i=1}^{N}\sum_{k_i} y_{2k_i}(\infty)|\cdots g_2^i \cdots\rangle|1_{k_i}\rangle \\
&+ \sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}} Z_{2k_ik_{ii}}(\infty)|\cdots g_2^i \cdots g_2^{ii} \cdots\rangle|1_{k_i}1_{k_{ii}}\rangle \\
&+ \sum_{i=1}^{N}\sum_{k_i}\sum_{q_i} R_{1k_iq_i}(\infty)|\cdots g_1^i \cdots\rangle|1_{k_i}\rangle|1_{q_i}\rangle \\
&+ \sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}}\sum_{q_i} T_{6k_ik_{ii}q_i}(\infty) \\
&\times |\cdots g_2^i \cdots g_1^{ii} \cdots\rangle|1_{k_i}1_{k_{ii}}\rangle|1_{q_i}\rangle \\
&+ \sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}}\sum_{q_i}\sum_{q_{ii}} T_{12k_ik_{ii}q_iq_{ii}}(\infty) \\
&\times |\cdots g_1^i \cdots g_1^{ii} \cdots\rangle|1_{k_i}1_{k_{ii}}\rangle|1_{q_i}1_{q_{ii}}\rangle. \qquad (19)
\end{aligned}$$

Compared with the result (14), the NSOC in this new system has a simpler form,

$$g_{1,2}^{(2)}(t,t) = \frac{p_1}{p_2 p_3}, \qquad (20a)$$

$$p_1 = \sum_{i=1}^{N}\sum_{k_i}\sum_{q_i}|R_{1k_iq_i}(\infty)|^2 + 2\sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}}\sum_{q_i}|T_{6k_ik_{ii}q_i}(\infty)|^2 + 8\sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}}\sum_{q_i}\sum_{q_{ii}}|T_{12k_ik_{ii}q_iq_{ii}}(\infty)|^2, \quad (20b)$$

$$p_2 = \sum_{i=1}^{N}\sum_{k_i}|y_{2k_i}(\infty)|^2 + \sum_{i=1}^{N}\sum_{k_i}\sum_{q_i}|R_{1k_iq_i}(\infty)|^2 + 4\sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}}|Z_{2k_ik_{ii}}(\infty)|^2 + 2\sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}}\sum_{q_i}|T_{6k_ik_{ii}q_i}(\infty)|^2 + 4\sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}}\sum_{q_i}\sum_{q_{ii}}|T_{12k_ik_{ii}q_iq_{ii}}(\infty)|^2, \quad (20c)$$

$$p_3 = \sum_{i=1}^{N}\sum_{k_i}\sum_{q_i}|R_{1k_iq_i}(\infty)|^2 + \sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}}\sum_{q_i}|T_{6k_ik_{ii}q_i}(\infty)|^2 + 4\sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}}\sum_{q_i}\sum_{q_{ii}}|T_{12k_ik_{ii}q_iq_{ii}}(\infty)|^2. \quad (20d)$$

The dependence of NSOC on the two decay rates γ_1' and γ_2' in this new scheme is plotted in Figure 5, where short T_1, quite long T_2 and suitable $T_3 \approx \frac{\pi}{2|\Omega_2|}$ are set. It is found that large γ_1' and small γ_2' are favor the quantum-mechanical correlation between the two emission fields [Figs. 5(a1) and 5(b1)]. Fast decay $|e_1\rangle \rightarrow |g_2\rangle$ (γ_1') increases the population in $|g_2\rangle$ at the end of the first scattering, and thus enhance the coherence of the two lower levels $|g_1\rangle$ and $|g_2\rangle$. Because the duration $T_3 \approx \frac{\pi}{2|\Omega_2|}$ is set for the *write* pulse (with Rabi frequency Ω_2 and detuning $\Delta_2 = 0$), the population in level $|g_2\rangle$ will completely transfer to level $|e_2\rangle$. The increasing decay rate γ_2' ($|e_2\rangle \rightarrow |g_1\rangle$) causes the decreasing of the population in level $|e_2\rangle$ at the end of pumping, and accordingly decrease the correlation between the final k-mode and q-mode fields.

Although the decay $|e_1\rangle \rightarrow |g_1\rangle$ are neglected here, our intuition tells us that it will reduce the correlation in real system. In fact, we can remove its influence by choosing a suitable pumping time [see Fig. 6]. If the decays $|e_1\rangle \rightarrow |g_2\rangle$ and $|e_2\rangle \rightarrow |g_1\rangle$ during the two pumpings (processes 1 and 3) are neglected in this new scheme (same to the first system by ignoring the decays $|e_1\rangle \rightarrow |g_1\rangle$ and $|e_2\rangle \rightarrow |g_2\rangle$) and the durations T_2 and T_4 are long enough ($T_2 = 20.0/\gamma_1'$ and $T_4 = \infty$ in our calculation), then the decay from $|e_1\rangle$ (or $|e_2\rangle$) to $|g_2\rangle$ (or $|g_1\rangle$) of all excited atoms will be complete in the process 2 (or 4), and the correlation function (NSOC) of the total field will be controlled entirely by the external *write* and *read* pulses and has nothing to do with the decay rate γ_1' and γ_2' [see Figs. 5(a2) and 5(b2)].

Now we can make a comparison between the two situations: including decays during pumping or not. It is shown that there should be no obvious difference between these two situations if the two decays $|e_1\rangle \rightarrow |g_2\rangle$ and $|e_2\rangle \rightarrow |g_1\rangle$ are slow (for example, $\gamma_1'\langle 10|\Omega_1|$ and $\gamma_2'\langle 0.5|\Omega_2|$ according to Fig. 5). In Figure 2, we set $\gamma_1' = 2|\Omega_1|$ and $\gamma_2' = 0.02|\Omega_2|$, and hence the approximation of ignoring the decays during pumping in our previous system is acceptable.

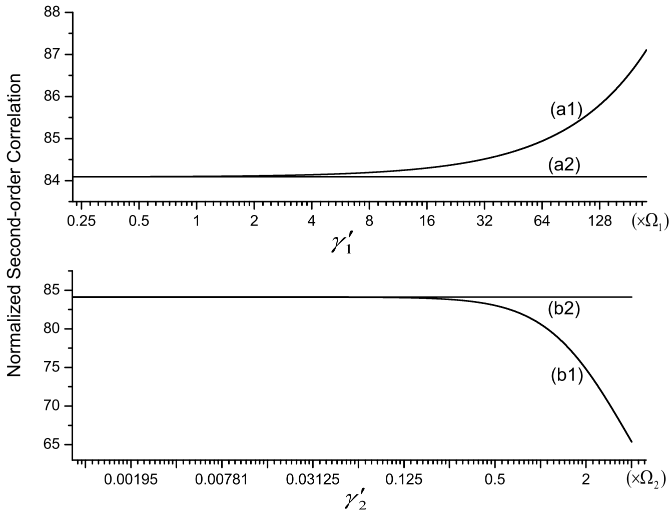

Fig. 5. Relations of NSOC with the decay rates γ_1' and γ_2' in the second scheme, for $N = 50\,000$, $\Delta_1 = \Delta_2 = 0$, $\Omega_2 = 100\Omega_1$, $T_1 = 0.001/\Omega_1$, $T_2 = 20.0/\gamma_1'$ and $T_3 = 1.57/\Omega_2$. (a1) $\gamma_2' = 2\Omega_1$, (a2) $\gamma_2' = 2\Omega_1$ and the decays during pumping are neglected; (b1) $\gamma_1' = 2\Omega_1$, (b2) $\gamma_1' = 2\Omega_1$ and the decays during pumping are neglected.

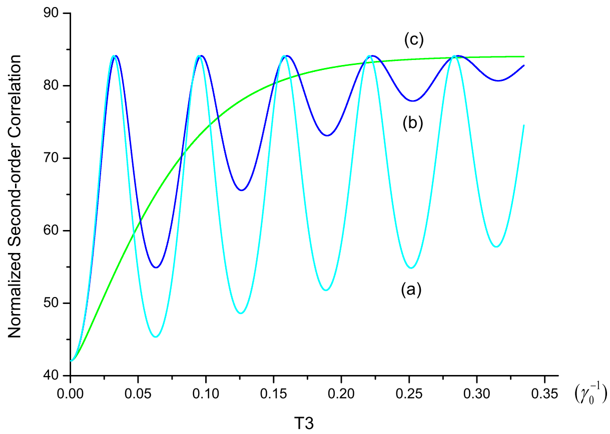

Fig. 6. Evolutions of NSOC versus the duration of read pulse T_3, for $N = 50\,000$, $\Delta_1 = \Delta_2 = 0$, $\gamma_1' = \gamma_0$, $\Omega_1 = 0.5\gamma_0$, $\Omega_2 = 50\gamma_0$, $T_1 = 0.001/\Omega_1$ and $T_2 = 20.0/\gamma_0$, (a) $\gamma_2' = 5.0\gamma_0$, (b) $\gamma_2' = 20.0\gamma_0$, (c) $\gamma_2' = 500.0\gamma_0 > (4\Omega_2)$.

The decay $|e_2\rangle \rightarrow |g_1\rangle$ during the second pumping process also changes the NSOC's dependence on durations T_3 [Fig. 6]. In resonance condition ($\Delta_2 = 0$), the NSOC oscillates with the duration T_3 when $\gamma_2'\langle 4|\Omega_2|$. The smaller the decay rate γ_2' is, the faster the oscillation will be. If the decay is strong enough ($\gamma_2'\rangle 4|\Omega_2|$), the oscillation disappears. In fact, the final state depends on T_3 through the exponential functions $e^{\lambda_3 T_3}$, $e^{\lambda_4 T_3}$, $e^{\frac{1}{2}\gamma_1' T_3}$, $e^{\frac{1}{2}\gamma_2' T_3}$ and their squares (please refer to Appendix B). The decay rate γ_1' and γ_2' are real numbers, and the parameters λ_3 and λ_4 are controlled by the *read* pulse's Rabi frequency Ω_2 and detuning Δ_2 through relation $\lambda^2 + (\frac{1}{2}\gamma_2' - i\Delta_2)\lambda + |\Omega_2|^2 = 0$, which tells us that: at resonance, if $\gamma_2'\langle 4|\Omega_2|$, λ_3 and λ_4 are complex numbers, then the state vector of the system oscillates with increasing T_3; otherwise λ_3 and λ_4 are real numbers and no oscillation appears.

5 Conclusion

We investigated the generation of two quantum-mechanically correlated thermal fields emitted by a four-level atomic ensemble. The use of the four-level energy structure has some advantages. The emissions of the *write* and *read* signal photons are associated with different upper levels, so that the emission processes are separable in time domain, even for limited coherence time; The two external laser pulses pump the atomic system from different lower levels to different upper levels, which makes the system easier to control; In three-level atomic system, the separation of the *write* and *read* signals are mainly through their orthogonal polarizations. In the four-level system, we can add the frequency difference to distinguish the signals and to improve the efficiency. In the scheme, the atomic coherence between the two lower levels plays an important role in establishing and enhancing the quantum correlation between the emission fields. The coherence is established by two lasers, which generate the coherence between $|e_1\rangle$ or $|e_2\rangle$ and $|g_1\rangle$, and the decay from $|e_1\rangle$ to $|g_1\rangle$, which generates the coherence between the two lower levels. To obtain strong correlation, the population in level $|g_2\rangle$ needs to be pumped to level $|e_2\rangle$ as much as possible, which could be realized by appropriately adjusting the pumping time T_3. The controllable interval between the two external laser pulses determines the time delay of the two emission fields. The delay will preserve rather than destroy the quantum correlation during the coherence time of the two lower levels, which is useful in some potential applications, such as quantum memory. If the decay rates are much smaller than the Rabi frequency of the driving field (for example: $\gamma_1 < 10|\Omega_1|$ in the first pumping and $\gamma_2 < 0.5|\Omega_2|$ in the second pumping), their influences in the pumping processes are insignificant and can be ignored. In addition, all the discussions presented here are based on the assumption that all the atoms are confined in a very small area compared with the wave lengths of the pumping lasers and emission fields, and the case of ensemble with large size will be discussed in the future.

The non-classical correlation expected in current atomic system is able to be verified through a similar setup used in references [23] and [24], except the four-level atomic ensemble. Just as demonstrated in reference [25], ^{87}Rb atoms may be a good choice for the atomic ensemble for the current scheme, with the hyperfine levels $F = 1$ and $F = 2$ of the ground state $5S_{1/2}$ serving as the two lower levels $|g_1\rangle$ and $|g_2\rangle$, and the hyperfine levels $F' = 2$ of $5P_{1/2}$ and $F' = 2$ of $5P_{3/2}$ serving as the two upper states $|e_1\rangle$ and $|e_2\rangle$. The four-level energy structure with two upper Zeeman sublevels and two lower Zeeman sublevels can also be found in other atoms, such as a $J = 1/2$ to $J = 1/2$ transition of $^{198}Hg^+$ ions [31]. Recently, Felinto et al. proposed an experimental technique to improve the coherence time in an atomic ensemble. An increase of two orders of magnitude longer than the previous results has been reported [32–34]. It may be a good attempt to apply this technique into the scheme proposed here.

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Appendix A: Calculation of the state vector in the last process of the first system

In the last process [see Fig. 1(4)], the exited state $|e_1\rangle$ decays to the two lower levels through channels $|e_1\rangle \to |g_1\rangle$ and $|e_1\rangle \to |g_2\rangle$, with decay rates γ_1 and γ_1' respectively. The exited state $|e_2\rangle$ are linked to the two lower levels through channels $|e_2\rangle \to |g_1\rangle$ and $|e_2\rangle \to |g_2\rangle$, with decay rates γ_2' and γ_2. No other interactions exist. The rotating wave-approximation Hamiltonian in this process is:

$$\begin{aligned}V_4 = {} & i\sum_{j=1}^{N}\sum_{k_j} g_{k_j}^{*} e^{i\boldsymbol{k}_j\cdot\boldsymbol{r}_j} e^{i(\omega_1-\omega_{k_j})t} a_{k_j}|e_1^j\rangle\langle g_1^j| + i\sum_{j=1}^{N}\sum_{k_j} g_{k_j}'^{*} e^{i\boldsymbol{k}_j\cdot\boldsymbol{r}_j} e^{i(\omega_1-\omega_{k_j})t} a_{k_j}|e_1^j\rangle\langle g_2^j| \\ & + i\sum_{j=1}^{N}\sum_{q_j} g_{q_j}^{*} e^{i\boldsymbol{q}_j\cdot\boldsymbol{r}_j} e^{i(\omega_2-\omega_{q_j})t} a_{q_j}|e_2^j\rangle\langle g_2^j| + i\sum_{j=1}^{N}\sum_{q_j} g_{q_j}'^{*} e^{i\boldsymbol{q}_j\cdot\boldsymbol{r}_j} e^{i(\omega_2-\omega_{q_j})t} a_{q_j}|e_2^j\rangle\langle g_1^j| + h.c. \end{aligned} \tag{A.1}$$

The initial state of this process is the final state of the previous process (duration T_3 is assumed for the third process):

$$\begin{aligned}|\phi_4(0)\rangle = |\phi_3(T_3)\rangle = {} & x_0^{(4)}(0)|g_1^1\cdots g_1^N\rangle|0\rangle + \sum_{i=1}^{N} x_i^{(4)}(0)|\cdots e_1^i\cdots\rangle|0\rangle + \cdots \\ & + \sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}} T_{8k_ik_{ii}}^{(4)}(0)|\cdots e_2^i\cdots e_2^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle. \end{aligned} \tag{A.2}$$

We assume that the state vector at time t in this process has the form like this:

$$\begin{aligned}|\phi_4(t)\rangle = {} & x_0^{(4)}(t)|g_1^1\cdots g_1^N\rangle|0\rangle + \sum_{i=1}^{N} x_i^{(4)}(t)|\cdots e_1^i\cdots\rangle|0\rangle + \sum_{i=1}^{N}\sum_{ii\neq i}^{N} x_{i,ii}^{(4)}(t)|\cdots e_1^i\cdots e_1^{ii}\cdots\rangle|0\rangle + \sum_{i=1}^{N}\sum_{k_i} y_{1k_i}^{(4)}(t)|\cdots g_1^i\cdots\rangle|1_{k_i}\rangle \\
& + \sum_{i=1}^{N}\sum_{k_i} y_{2k_i}^{(4)}(t)|\cdots g_2^i\cdots\rangle|1_{k_i}\rangle + \sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i} y_{3k_i,ii}^{(4)}(t)|\cdots g_1^i\cdots e_1^{ii}\cdots\rangle|1_{k_i}\rangle + \sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i} y_{4k_i,ii}^{(4)}(t)|\cdots g_2^i\cdots e_1^{ii}\cdots\rangle|1_{k_i}\rangle \\
& + \sum_{i=1}^{N}\sum_{k_i} R_{0k_i}^{(4)}(t)|\cdots e_2^i\cdots\rangle|1_{k_i}\rangle + \sum_{i=1}^{N}\sum_{k_i}\sum_{q_i} R_{1k_iq_i}^{(4)}(t)|\cdots g_1^i\cdots\rangle|1_{k_i}\rangle|1_{q_i}\rangle + \sum_{i=1}^{N}\sum_{k_i}\sum_{q_i} R_{2k_iq_i}^{(4)}(t)|\cdots g_2^i\cdots\rangle|1_{k_i}\rangle|1_{q_i}\rangle \\
& + \sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}} Z_{1k_ik_{ii}}^{(4)}(t)|\cdots g_1^i\cdots g_1^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle + \sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}} Z_{2k_ik_{ii}}^{(4)}(t)|\cdots g_2^i\cdots g_2^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle \\
& + \sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}} Z_{3k_ik_{ii}}^{(4)}(t)|\cdots g_1^i\cdots g_2^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle + \sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i} Z_{4k_i,ii}^{(4)}(t)|\cdots e_2^i\cdots e_1^{ii}\cdots\rangle|1_{k_i}\rangle \\
& + \sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}} T_{1k_ik_{ii}}^{(4)}(t)|\cdots e_2^i\cdots g_2^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}} + \sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}} T_{2k_ik_{ii}}^{(4)}(t)|\cdots e_2^i\cdots g_1^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle \\
& + \sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{q_i} T_{3k_iq_i,ii}^{(4)}(t)|\cdots g_1^i\cdots e_1^{ii}\cdots\rangle|1_{k_i}\rangle|1_{q_i}\rangle + \sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{q_i} T_{4k_iq_i,ii}^{(4)}(t) \\
& \times|\cdots g_2^i\cdots e_1^{ii}\cdots\rangle|1_{k_i}\rangle|1_{q_i}\rangle + \sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}}\sum_{q_i} T_{5k_ik_{ii}q_i}^{(4)}(t)|\cdots g_1^i\cdots g_1^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle|1_{q_i}\rangle \\
& + \sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}}\sum_{q_i} T_{6k_ik_{ii}q_i}^{(4)}(t)|\cdots g_1^i\cdots g_2^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle|1_{q_i}\rangle + \sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}}\sum_{q_i} T_{7k_ik_{ii}q_i}^{(4)}(t) \\
& \times|\cdots g_2^i\cdots g_2^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle|1_{q_i}\rangle + \sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}} T_{8k_ik_{ii}}^{(4)}(t)|\cdots e_2^i\cdots e_2^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle \\
& + \sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}}\sum_{q_i} T_{9k_ik_{ii}q_i}^{(4)}(t)|\cdots g_2^i\cdots e_2^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle|1_{q_i}\rangle + \sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}}\sum_{q_i} T_{10k_ik_{ii}q_i}^{(4)}(t) \\
& \times|\cdots g_1^i\cdots e_2^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle|1_{q_i}\rangle + \sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}}\sum_{q_i}\sum_{q_{ii}} T_{11k_ik_{ii}q_iq_{ii}}^{(4)}(t)\end{aligned}$$

$$\times|\cdots g_2^i\cdots g_2^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle|1_{q_i}1_{q_{ii}}\rangle+\sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}}\sum_{q_i}\sum_{q_{ii}}T^{(4)}_{12k_ik_{ii}q_iq_{ii}}(t)$$
$$\times|\cdots g_1^i\cdots g_1^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle|1_{q_i}1_{q_{ii}}\rangle+\sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}}\sum_{q_i}\sum_{q_{ii}}T^{(4)}_{13k_ik_{ii}q_iq_{ii}}(t)|\cdots g_1^i\cdots g_2^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle|1_{q_i}1_{q_{ii}}\rangle. \quad (A.3)$$

The Schrödinger equation gives the following relations:

$$\dot{x}_0^{(4)}(t)=0, \quad (A.4)$$

$$\dot{x}_i^{(4)}(t)=\sum_{k_i}g^*_{k_i}e^{i\boldsymbol{k}_i\cdot\boldsymbol{r}_i}e^{i(\omega_1-\omega_{k_i})t}y^{(4)}_{1k_i}(t)+\sum_{k_i}g'^*_{k_i}e^{i\boldsymbol{k}_i\cdot\boldsymbol{r}_i}e^{i(\omega_1-\omega_{k_i})t}y^{(4)}_{2k_i}(t), \quad (A.5a)$$
$$\dot{y}^{(4)}_{1k_i}(t)=-g_{k_i}e^{-i\boldsymbol{k}_i\cdot\boldsymbol{r}_i}e^{-i(\omega_1-\omega_{k_i})t}x_i^{(4)}(t), \quad (A.5b)$$
$$\dot{y}^{(4)}_{2k_i}(t)=-g'_{k_i}e^{-i\boldsymbol{k}_i\cdot\boldsymbol{r}_i}e^{-i(\omega_1-\omega_{k_i})t}x_i^{(4)}(t), \quad (A.5c)$$

$$\dot{x}^{(4)}_{i,ii}(t)=\sum_{k_i}g^*_{k_i}e^{i\boldsymbol{k}_i\cdot\boldsymbol{r}_i}e^{i(\omega_1-\omega_{k_i})t}y^{(4)}_{3k_i,ii}(t)+\sum_{k_i}g'^*_{k_i}e^{i\boldsymbol{k}_i\cdot\boldsymbol{r}_i}e^{i(\omega_1-\omega_{k_i})t}y^{(4)}_{4k_i,ii}(t), \quad (A.6a)$$
$$\dot{y}^{(4)}_{3k_i,ii}(t)=\sum_{k_{ii}}g^*_{k_{ii}}e^{i\boldsymbol{k}_{ii}\cdot\boldsymbol{r}_{ii}}e^{i(\omega_1-\omega_{k_{ii}})t}Z^{(4)}_{1k_ik_{ii}}(t)+\sum_{k_{ii}}g'^*_{k_{ii}}e^{i\boldsymbol{k}_{ii}\cdot\boldsymbol{r}_{ii}}e^{i(\omega_1-\omega_{k_{ii}})t}Z^{(4)}_{3k_ik_{ii}}(t)-2g_{k_i}e^{-i\boldsymbol{k}_i\cdot\boldsymbol{r}_i}e^{-i(\omega_1-\omega_{k_i})t}x^{(4)}_{i,ii}(t), \quad (A.6b)$$
$$\dot{y}^{(4)}_{4k_i,ii}(t)=\sum_{k_{ii}}g'^*_{k_{ii}}e^{i\boldsymbol{k}_{ii}\cdot\boldsymbol{r}_{ii}}e^{i(\omega_1-\omega_{k_{ii}})t}Z^{(4)}_{2k_ik_{ii}}(t)+\sum_{k_{ii}}g^*_{k_{ii}}e^{i\boldsymbol{k}_{ii}\cdot\boldsymbol{r}_{ii}}e^{i(\omega_1-\omega_{k_{ii}})t}Z^{(4)}_{3k_ik_{ii}}(t)-2g'_{k_i}e^{-i\boldsymbol{k}_i\cdot\boldsymbol{r}_i}e^{-i(\omega_1-\omega_{k_i})t}x^{(4)}_{i,ii}(t), \quad (A.6c)$$
$$\dot{Z}^{(4)}_{1k_ik_{ii}}(t)=-g_{k_{ii}}e^{-i\boldsymbol{k}_{ii}\cdot\boldsymbol{r}_{ii}}e^{-i(\omega_1-\omega_{k_{ii}})t}y^{(4)}_{3k_i,ii}(t), \quad (A.6d)$$
$$\dot{Z}^{(4)}_{2k_ik_{ii}}(t)=-g'_{k_{ii}}e^{-i\boldsymbol{k}_{ii}\cdot\boldsymbol{r}_{ii}}e^{-i(\omega_1-\omega_{k_{ii}})t}y^{(4)}_{4k_i,ii}(t), \quad (A.6e)$$
$$\dot{Z}^{(4)}_{3k_ik_{ii}}(t)=-g_{k_{ii}}e^{-i\boldsymbol{k}_{ii}\cdot\boldsymbol{r}_{ii}}e^{-i(\omega_1-\omega_{k_{ii}})t}y^{(4)}_{4k_i,ii}(t)-g'_{k_{ii}}e^{-i\boldsymbol{k}_{ii}\cdot\boldsymbol{r}_{ii}}e^{-i(\omega_1-\omega_{k_{ii}})t}y^{(4)}_{3k_i,ii}(t), \quad (A.6f)$$

$$\dot{R}^{(4)}_{0k_i}(t)=\sum_{q_i}g'^*_{q_i}e^{i\boldsymbol{q}_i\cdot\boldsymbol{r}_i}e^{i(\omega_2-\omega_{q_i})t}R^{(4)}_{1k_iq_i}(t)+\sum_{q_i}g^*_{q_i}e^{i\boldsymbol{q}_i\cdot\boldsymbol{r}_i}e^{i(\omega_2-\omega_{q_i})t}R^{(4)}_{2k_iq_i}(t), \quad (A.7a)$$
$$\dot{R}^{(4)}_{1k_iq_i}(t)=-g'_{q_i}e^{-i\boldsymbol{q}_i\cdot\boldsymbol{r}_i}e^{-i(\omega_2-\omega_{q_i})t}R^{(4)}_{0k_i}(t), \quad (A.7b)$$
$$\dot{R}^{(4)}_{2k_iq_i}(t)=-g_{q_i}e^{-i\boldsymbol{q}_i\cdot\boldsymbol{r}_i}e^{-i(\omega_2-\omega_{q_i})t}R^{(4)}_{0k_i}(t), \quad (A.7c)$$

$$\begin{aligned}\dot{Z}^{(4)}_{4k_i,ii}(t)=&\sum_{k_{ii}}g'^*_{k_{ii}}e^{i\boldsymbol{k}_{ii}\cdot\boldsymbol{r}_{ii}}e^{i(\omega_1-\omega_{k_{ii}})t}T^{(4)}_{1k_ik_{ii}}(t)+\sum_{k_{ii}}g^*_{k_{ii}}e^{i\boldsymbol{k}_{ii}\cdot\boldsymbol{r}_{ii}}e^{i(\omega_1-\omega_{k_{ii}})t}T^{(4)}_{2k_ik_{ii}}(t)\\&+\sum_{q_i}g'^*_{q_i}e^{i\boldsymbol{q}_i\cdot\boldsymbol{r}_i}e^{i(\omega_2-\omega_{q_i})t}T^{(4)}_{3k_iq_i,ii}(t)+\sum_{q_i}g^*_{q_i}e^{i\boldsymbol{q}_i\cdot\boldsymbol{r}_i}e^{i(\omega_2-\omega_{q_i})t}T^{(4)}_{4k_iq_i,ii}(t),\end{aligned} \quad (A.8a)$$
$$\begin{aligned}\dot{T}^{(4)}_{1k_ik_{ii}}(t)=&\sum_{q_i}g'^*_{q_i}e^{i\boldsymbol{q}_i\cdot\boldsymbol{r}_i}e^{i(\omega_2-\omega_{q_i})t}T^{(4)}_{6k_ik_{ii}q_i}(t)+\sum_{q_i}g^*_{q_i}e^{i\boldsymbol{q}_i\cdot\boldsymbol{r}_i}e^{i(\omega_2-\omega_{q_i})t}T^{(4)}_{7k_ik_{ii}q_i}(t)\\&-g'_{k_{ii}}e^{-i\boldsymbol{k}_{ii}\cdot\boldsymbol{r}_{ii}}e^{-i(\omega_1-\omega_{k_{ii}})t}Z^{(4)}_{4k_i,ii}(t),\end{aligned} \quad (A.8b)$$
$$\begin{aligned}\dot{T}^{(4)}_{2k_ik_{ii}}(t)=&\sum_{q_i}g'^*_{q_i}e^{i\boldsymbol{q}_i\cdot\boldsymbol{r}_i}e^{i(\omega_2-\omega_{q_i})t}T^{(4)}_{5k_ik_{ii}q_i}(t)+\sum_{q_i}g^*_{q_i}e^{i\boldsymbol{q}_i\cdot\boldsymbol{r}_i}e^{i(\omega_2-\omega_{q_i})t}T^{(4)}_{6k_ik_{ii}q_i}(t)\\&-g_{k_{ii}}e^{-i\boldsymbol{k}_{ii}\cdot\boldsymbol{r}_{ii}}e^{-i(\omega_1-\omega_{k_{ii}})t}Z^{(4)}_{4k_i,ii}(t),\end{aligned} \quad (A.8c)$$

$$\dot{T}^{(4)}_{3k_iq_i,ii}(t)=\sum_{k_{ii}}g^*_{k_{ii}}e^{i\boldsymbol{k}_{ii}\cdot\boldsymbol{r}_{ii}}e^{i(\omega_1-\omega_{k_{ii}})t}T^{(4)}_{5k_ik_{ii}q_i}(t)+\sum_{k_{ii}}g'^*_{k_{ii}}e^{i\boldsymbol{k}_{ii}\cdot\boldsymbol{r}_{ii}}e^{i(\omega_1-\omega_{k_{ii}})t}T^{(4)}_{6k_ik_{ii}q_i}(t)$$
$$-g'_{q_i}e^{-i\boldsymbol{q}_i\cdot\boldsymbol{r}_i}e^{-i(\omega_2-\omega_{q_i})t}Z^{(4)}_{4k_i,ii}(t),\tag{A.8d}$$

$$\dot{T}^{(4)}_{4k_iq_i,ii}(t)=\sum_{k_{ii}}g^*_{k_{ii}}e^{i\boldsymbol{k}_{ii}\cdot\boldsymbol{r}_{ii}}e^{i(\omega_1-\omega_{k_{ii}})t}T^{(4)}_{6k_ik_{ii}q_i}(t)+\sum_{k_{ii}}g'^*_{k_{ii}}e^{i\boldsymbol{k}_{ii}\cdot\boldsymbol{r}_{ii}}e^{i(\omega_1-\omega_{k_{ii}})t}T^{(4)}_{7k_ik_{ii}q_i}(t)$$
$$-g_{q_i}e^{-i\boldsymbol{q}_i\cdot\boldsymbol{r}_i}e^{-i(\omega_2-\omega_{q_i})t}Z^{(4)}_{4k_i,ii}(t),\tag{A.8e}$$

$$\dot{T}^{(4)}_{5k_ik_{ii}q_i}(t)=-g'_{q_i}e^{-i\boldsymbol{q}_i\cdot\boldsymbol{r}_i}e^{-i(\omega_2-\omega_{q_i})t}T^{(4)}_{2k_ik_{ii}}(t)-g_{k_{ii}}e^{-i\boldsymbol{k}_{ii}\cdot\boldsymbol{r}_{ii}}e^{-i(\omega_1-\omega_{k_{ii}})t}T^{(4)}_{3k_iq_i,ii}(t),\tag{A.8f}$$

$$\dot{T}^{(4)}_{6k_ik_{ii}q_i}(t)=-g'_{q_i}e^{-i\boldsymbol{q}_i\cdot\boldsymbol{r}_i}e^{-i(\omega_2-\omega_{q_i})t}T^{(4)}_{1k_ik_{ii}}(t)-g_{q_i}e^{-i\boldsymbol{q}_i\cdot\boldsymbol{r}_i}e^{-i(\omega_2-\omega_{q_i})t}T^{(4)}_{2k_ik_{ii}}(t)$$
$$-g'_{k_{ii}}e^{-i\boldsymbol{k}_{ii}\cdot\boldsymbol{r}_{ii}}e^{-i(\omega_1-\omega_{k_{ii}})t}T^{(4)}_{3k_iq_i,ii}(t)-g_{k_{ii}}e^{-i\boldsymbol{k}_{ii}\cdot\boldsymbol{r}_{ii}}e^{-i(\omega_1-\omega_{k_{ii}})t}T^{(4)}_{4k_iq_i,ii}(t),\tag{A.8g}$$

$$\dot{T}^{(4)}_{7k_ik_{ii}q_i}(t)=-g'_{q_i}e^{-i\boldsymbol{q}_i\cdot\boldsymbol{r}_i}e^{-i(\omega_2-\omega_{q_i})t}T^{(4)}_{1k_ik_{ii}}(t)-g_{k_{ii}}e^{-i\boldsymbol{k}_{ii}\cdot\boldsymbol{r}_{ii}}e^{-i(\omega_1-\omega_{k_{ii}})t}T^{(4)}_{4k_iq_i,ii}(t),\tag{A.8h}$$

$$\dot{T}^{(4)}_{8k_ik_{ii}}(t)=\sum_{q_i}g^*_{q_i}e^{i\boldsymbol{q}_i\cdot\boldsymbol{r}_i}e^{i(\omega_2-\omega_{q_i})t}T^{(4)}_{9k_iq_i,ii}(t)+\sum_{q_i}g'^*_{q_i}e^{i\boldsymbol{q}_i\cdot\boldsymbol{r}_i}e^{i(\omega_2-\omega_{q_i})t}T^{(4)}_{10k_ik_{ii}q_i}(t),\tag{A.9a}$$

$$\dot{T}^{(4)}_{9k_ik_{ii}q_i}(t)=\sum_{q_{ii}}g^*_{q_{ii}}e^{i\boldsymbol{q}_{ii}\cdot\boldsymbol{r}_{ii}}e^{i(\omega_2-\omega_{q_{ii}})t}T^{(4)}_{11k_ik_{ii}q_iq_{ii}}(t)+\sum_{q_{ii}}g'^*_{q_{ii}}e^{i\boldsymbol{q}_{ii}\cdot\boldsymbol{r}_{ii}}e^{i(\omega_2-\omega_{q_{ii}})t}T^{(4)}_{13k_ik_{ii}q_iq_{ii}}(t)$$
$$-2g_{q_i}e^{-i\boldsymbol{q}_i\cdot\boldsymbol{r}_i}e^{-i(\omega_2-\omega_{q_i})t}T^{(4)}_{8k_ik_{ii}}(t),\tag{A.9b}$$

$$\dot{T}^{(4)}_{10k_ik_{ii}q_i}(t)=\sum_{q_{ii}}g^*_{q_{ii}}e^{i\boldsymbol{q}_{ii}\cdot\boldsymbol{r}_{ii}}e^{i(\omega_2-\omega_{q_{ii}})t}T^{(4)}_{13k_ik_{ii}q_iq_{ii}}(t)+\sum_{q_{ii}}g'^*_{q_{ii}}e^{i\boldsymbol{q}_{ii}\cdot\boldsymbol{r}_{ii}}e^{i(\omega_2-\omega_{q_{ii}})t}T^{(4)}_{12k_ik_{ii}q_iq_{ii}}(t)$$
$$-2g'_{q_i}e^{-i\boldsymbol{q}_i\cdot\boldsymbol{r}_i}e^{-i(\omega_2-\omega_{q_i})t}T^{(4)}_{8k_ik_{ii}}(t),\tag{A.9c}$$

$$\dot{T}^{(4)}_{11k_ik_{ii}q_iq_{ii}}(t)=-g_{q_{ii}}e^{-i\boldsymbol{q}_{ii}\cdot\boldsymbol{r}_{ii}}e^{-i(\omega_2-\omega_{q_{ii}})t}T^{(4)}_{9k_ik_{ii}q_i}(t),\tag{A.9d}$$

$$\dot{T}^{(4)}_{12k_ik_{ii}q_iq_{ii}}(t)=-g'_{q_{ii}}e^{-i\boldsymbol{q}_{ii}\cdot\boldsymbol{r}_{ii}}e^{-i(\omega_2-\omega_{q_{ii}})t}T^{(4)}_{10k_ik_{ii}q_i}(t),\tag{A.9e}$$

$$\dot{T}^{(4)}_{13k_ik_{ii}q_iq_{ii}}(t)=-g'_{q_{ii}}e^{-i\boldsymbol{q}_{ii}\cdot\boldsymbol{r}_{ii}}e^{-i(\omega_2-\omega_{q_{ii}})t}T^{(4)}_{9k_ik_{ii}q_i}(t)-g_{q_{ii}}e^{-i\boldsymbol{q}_{ii}\cdot\boldsymbol{r}_{ii}}e^{-i(\omega_2-\omega_{q_{ii}})t}T^{(4)}_{10k_ik_{ii}q_i}(t).\tag{A.9f}$$

The equations in groups (A.4) and (A.5) can be easily solved,

$$x^{(4)}_0(t)=x^{(4)}_0(0),\tag{A.10}$$

$$x^{(4)}_i(t)=x^{(4)}_i(0)e^{-\frac{1}{2}(\gamma_1+\gamma'_1)t}.\tag{A.11}$$

By integrating the equations (A.6d), (A.6e) and (A.6f) and substituting the results into equations (A.6b) and (A.6c), we get:

$$\dot{y}^{(4)}_{3k_i,ii}(t)=-2g_{k_i}e^{-i\boldsymbol{k}_i\cdot\boldsymbol{r}_0}e^{-i(\omega_1-\omega_{k_i})t}x^{(4)}_{i,ii}(t)-\frac{1}{2}(\gamma_1+\gamma'_1)y^{(4)}_{3k_i,ii}(t)-\frac{1}{2}p_1\sqrt{\gamma_1\gamma'_1}y^{(4)}_{4k_i,ii}(t),\tag{A.12a}$$

$$\dot{y}^{(4)}_{4k_i,ii}(t)=-2g'_{k_i}e^{-i\boldsymbol{k}_i\cdot\boldsymbol{r}_0}e^{-i(\omega_1-\omega_{k_i})t}x^{(4)}_{i,ii}(t)-\frac{1}{2}(\gamma_1+\gamma'_1)y^{(4)}_{4k_i,ii}(t)-\frac{1}{2}p_1\sqrt{\gamma_1\gamma'_1}y^{(4)}_{3k_i,ii}(t).\tag{A.12b}$$

p_1 is the alignment of the two dipole moments of transitions $|e_1\rangle\to|g_1\rangle$ and $|e_1\rangle\to|g_2\rangle$, associated with the interference of the k-mode photons emitted from different atoms. Here we consider the simple situation without interference, i.e., $p_1=0$. From equations (A.12a) and (A.12b), we have:

$$y^{(4)}_{3k_i,ii}(t)=-2e^{-\frac{1}{2}(\gamma_1+\gamma'_1)t}\int_0^t g_{k_i}e^{-i\boldsymbol{k}_i\cdot\boldsymbol{r}_0}e^{-i(\omega_1-\omega_{k_i})t'}e^{\frac{1}{2}(\gamma_1+\gamma'_1)t'}x^{(4)}_{i,ii}(t')dt',\tag{A.13a}$$

$$y^{(4)}_{4k_i,ii}(t)=-2e^{-\frac{1}{2}(\gamma_1+\gamma'_1)t}\int_0^t g'_{k_i}e^{-i\boldsymbol{k}_i\cdot\boldsymbol{r}_0}e^{-i(\omega_1-\omega_{k_i})t'}e^{\frac{1}{2}(\gamma_1+\gamma'_1)t'}x^{(4)}_{i,ii}(t')dt'.\tag{A.13b}$$

Combining the equations (A.13a), (A.13b) and (A.6a), we obtain the differential equation for $x^{(4)}_{i,ii}(t)$,

$$\dot{x}^{(4)}_{i,ii}(t)=-(\gamma_1+\gamma'_1)x^{(4)}_{i,ii}(t),\tag{A.14a}$$

and so,

$$x^{(4)}_{i,ii}(t) = x^{(4)}_{i,ii}(0)e^{-(\gamma_1+\gamma_1')t}. \tag{A.14b}$$

The expressions for $y^{(4)}_{1k_i}(t)$, $y^{(4)}_{2k_i}(t)$, $y^{(4)}_{3k_i,ii}(t)$, $y^{(4)}_{4k_i,ii}(t)$ can be obtained directly via $x^{(4)}_i(t)$ and $x^{(4)}_{i,ii}(t)$ based on the relations (A.5b), (A.5c), (A.13a) and (A.13b). Similarly, the results of $Z^{(4)}_{1k_ikii}(t)$, $Z^{(4)}_{2k_ikii}(t)$ and $Z^{(4)}_{3k_ikii}(t)$ are also easy to obtain by integrating equations (A.6d), (A.6e) and (A.6f). The integral $\int_{-\infty}^{\infty} e^{i(\omega_1-\omega_k)(t-t')}d\omega_k = 2\pi\delta(t-t')$ and the symmetry $x^{(4)}_{i,ii}(t) = x^{(4)}_{ii,i}(t)$ is applied in our calculation.

The solutions of group (A.7) can be written down directly as:

$$R^{(4)}_{0k_i}(t) = R^{(4)}_{0k_i}(0)e^{-\frac{1}{2}(\gamma_2+\gamma_2')t}, \tag{A.15a}$$

$$R^{(4)}_{1k_iq_i}(t) = -R^{(4)}_{0k_i}(0)\int_0^t g'_{q_i}e^{-i\boldsymbol{q}_i\cdot\boldsymbol{r}_0}e^{-i(\omega_2-\omega_{q_i})t'}e^{-\frac{1}{2}(\gamma_2+\gamma_2')t'}dt', \tag{A.15b}$$

$$R^{(4)}_{2k_iq_i}(t) = -R^{(4)}_{0k_i}(0)\int_0^t g_{q_i}e^{-i\boldsymbol{q}_i\cdot\boldsymbol{r}_0}e^{-i(\omega_2-\omega_{q_i})t'}e^{-\frac{1}{2}(\gamma_2+\gamma_2')t'}dt'. \tag{A.15c}$$

By integrating the equations (A.8f), (A.8g) and (A.8h) and substituting the results into equations (A.8b), (A.8c), (A.8d) and (A.8e), we obtain:

$$\dot{T}^{(4)}_{1k_ik_{ii}}(t) = -g'_{k_{ii}}e^{-i\boldsymbol{k}_{ii}\cdot\boldsymbol{r}_0}e^{-i(\omega_1-\omega_{k_{ii}})t}Z^{(4)}_{4k_i,ii}(t) - \frac{1}{2}(\gamma_2+\gamma_2')T^{(4)}_{1k_ik_{ii}}(t) - \frac{1}{2}p_2\sqrt{\gamma_2\gamma_2'}T^{(4)}_{2k_ik_{ii}}(t), \tag{A.16a}$$

$$\dot{T}^{(4)}_{2k_ik_{ii}}(t) = -g_{k_{ii}}e^{-i\boldsymbol{k}_{ii}\cdot\boldsymbol{r}_0}e^{-i(\omega_1-\omega_{k_{ii}})t}Z^{(4)}_{4k_i,ii}(t) - \frac{1}{2}(\gamma_2+\gamma_2')T^{(4)}_{2k_ik_{ii}}(t) - \frac{1}{2}p_2\sqrt{\gamma_2\gamma_2'}T^{(4)}_{1k_ik_{ii}}(t), \tag{A.16b}$$

$$\dot{T}^{(4)}_{3k_iq_i,ii}(t) = -g'_{q_i}e^{-i\boldsymbol{q}_i\cdot\boldsymbol{r}_0}e^{-i(\omega_2-\omega_{q_i})t}Z^{(4)}_{4k_i,ii}(t) - \frac{1}{2}(\gamma_1+\gamma_1')T^{(4)}_{3k_iq_i,ii}(t) - \frac{1}{2}p_1\sqrt{\gamma_1\gamma_1'}T^{(4)}_{4k_ik_{ii}}(t), \tag{A.16c}$$

$$\dot{T}^{(4)}_{4k_iq_i,ii}(t) = -g_{q_i}e^{-i\boldsymbol{q}_i\cdot\boldsymbol{r}_0}e^{-i(\omega_2-\omega_{q_i})t}Z^{(4)}_{4k_i,ii}(t) - \frac{1}{2}(\gamma_1+\gamma_1')T^{(4)}_{4k_iq_i,ii}(t) - \frac{1}{2}p_1\sqrt{\gamma_1\gamma_1'}T^{(4)}_{3k_ik_{ii}}(t). \tag{A.16d}$$

p_2 is the alignment of the two dipole moments of transitions $|e_2\rangle \to |g_1\rangle$ and $|e_2\rangle \to |g_2\rangle$, associated with the interference of the q-mode photons emitted from different atoms. Here we consider the simple situation without interference, i.e., $p_2 = 0$. From equations in group (A.16), we get:

$$T^{(4)}_{1k_ik_{ii}}(t) = -e^{-\frac{1}{2}(\gamma_2+\gamma_2')t}\int_0^t g'_{k_{ii}}e^{-i\boldsymbol{k}_{ii}\cdot\boldsymbol{r}_0}e^{-i(\omega_1-\omega_{k_{ii}})t'}e^{\frac{1}{2}(\gamma_2+\gamma_2')t'}Z^{(4)}_{4k_i,ii}(t')dt', \tag{A.17a}$$

$$T^{(4)}_{2k_ik_{ii}}(t) = -e^{-\frac{1}{2}(\gamma_2+\gamma_2')t}\int_0^t g_{k_{ii}}e^{-i\boldsymbol{k}_{ii}\cdot\boldsymbol{r}_0}e^{-i(\omega_1-\omega_{k_{ii}})t'}e^{\frac{1}{2}(\gamma_2+\gamma_2')t'}Z^{(4)}_{4k_i,ii}(t')dt', \tag{A.17b}$$

$$T^{(4)}_{3k_iq_i,ii}(t) = -e^{-\frac{1}{2}(\gamma_1+\gamma_1')t}\int_0^t g'_{q_i}e^{-i\boldsymbol{q}_i\cdot\boldsymbol{r}_0}e^{-i(\omega_2-\omega_{q_i})t'}e^{\frac{1}{2}(\gamma_1+\gamma_1')t'}Z^{(4)}_{4k_i,ii}(t')dt', \tag{A.17c}$$

$$T^{(4)}_{4k_iq_i,ii}(t) = -e^{-\frac{1}{2}(\gamma_1+\gamma_1')t}\int_0^t g_{q_i}e^{-i\boldsymbol{q}_i\cdot\boldsymbol{r}_0}e^{-i(\omega_2-\omega_{q_i})t'}e^{\frac{1}{2}(\gamma_1+\gamma_1')t'}Z^{(4)}_{4k_i,ii}(t')dt'. \tag{A.17d}$$

Substituting group (A.17) into (A.8a), we obtain the differential equation for $Z^{(4)}_{4k_i,ii}(t)$,

$$\dot{Z}^{(4)}_{4k_i,ii}(t) = -\frac{1}{2}(\gamma_1+\gamma_1'+\gamma_2+\gamma_2')Z^{(4)}_{4k_i,ii}(t), \tag{A.18a}$$

and so,

$$Z^{(4)}_{4k_i,ii}(t) = Z^{(4)}_{4k_i,ii}(0)e^{-\frac{1}{2}(\gamma_1+\gamma_1'+\gamma_2+\gamma_2')t}. \tag{A.18b}$$

The expressions for other coefficients in group (A.8) can be obtained directly via $Z^{(4)}_{4k_i,ii}(t)$ (A.18b). By integrating the equations (A.9d), (A.9e) and (A.9f) and substituting the results into equations (A.9b) and (A.9c), we obtain:

$$\dot{T}^{(4)}_{9k_ik_{ii}q_i}(t) = -2g_{q_i}e^{-i\boldsymbol{q}_i\cdot\boldsymbol{r}_0}e^{-i(\omega_2-\omega_{q_i})t}T^{(4)}_{8k_ik_{ii}}(t) - \frac{1}{2}(\gamma_2+\gamma_2')T^{(4)}_{9k_ik_{ii}q_i}(t) - \frac{1}{2}p_2\sqrt{\gamma_2\gamma_2'}T^{(4)}_{10k_ik_{ii}q_i}(t), \tag{A.19a}$$

$$\dot{T}^{(4)}_{10k_ik_{ii}q_i}(t) = -2g'_{q_i}e^{-i\boldsymbol{q}_i\cdot\boldsymbol{r}_0}e^{-i(\omega_2-\omega_{q_i})t}T^{(4)}_{8k_ik_{ii}}(t) - \frac{1}{2}(\gamma_2+\gamma_2')T^{(4)}_{10k_ik_{ii}q_i}(t) - \frac{1}{2}p_2\sqrt{\gamma_2\gamma_2'}T^{(4)}_{9k_ik_{ii}q_i}(t). \tag{A.19b}$$

When $p_2 = 0$, the above two equations turn to,

$$T^{(4)}_{9k_ik_{ii}q_i}(t) = -2e^{-\frac{1}{2}(\gamma_2+\gamma_2')t}\int_0^t g_{q_i}e^{-i\boldsymbol{q}_i\cdot\boldsymbol{r}_0}e^{-i(\omega_2-\omega_{q_i})t'}e^{\frac{1}{2}(\gamma_2+\gamma_2')t'}T^{(4)}_{8k_ik_{ii}}(t')dt', \tag{A.19c}$$

$$T^{(4)}_{10k_ik_{ii}q_i}(t) = -2e^{-\frac{1}{2}(\gamma_2+\gamma_2')t}\int_0^t g'_{q_i}e^{-i\boldsymbol{q}_i\cdot\boldsymbol{r}_0}e^{-i(\omega_2-\omega_{q_i})t'}e^{\frac{1}{2}(\gamma_2+\gamma_2')t'}T^{(4)}_{8k_ik_{ii}}(t')dt'. \tag{A.19d}$$

From equations (A.19c), (A.19d) and (A.9a), we have:

$$\dot{T}^{(4)}_{8k_ik_{ii}}(t) = -(\gamma_2+\gamma_2')T^{(4)}_{8k_ik_{ii}}(t), \tag{A.20a}$$

and so,

$$T^{(4)}_{8k_ik_{ii}}(t) = T^{(4)}_{8k_ik_{ii}}(0)e^{-(\gamma_2+\gamma_2')t}. \tag{A.20b}$$

And other coefficients in group (A.9) can be solved directly according to relations (A.19c), (A.19d), (A.9d), (A.9e) and (A.9f).

Appendix B: Calculation of the state vector in the third process of the second system

In the third process [see Fig. 4(3)], the read pulse links the levels $|g_2\rangle$ and $|e_2\rangle$ with Rabi frequency Ω_2 and detuning Δ_2.The vacuum mode couples the exited states $|e_2\rangle$ and lower level $|g_1\rangle$. The rotating wave-approximation Hamiltonian in this process is:

$$\begin{aligned} V_3 = {} & i\sum_{j=1}^{N}\sum_{k_j} g^*_{k_j}e^{i\boldsymbol{k}_j\cdot\boldsymbol{r}_j}e^{i(\omega_1-\omega_{k_j})t}a_{k_j}|e_1^j\rangle\langle g_2^j| + i\sum_{j=1}^{N}\sum_{q_j} g^*_{q_j}e^{i\boldsymbol{q}_j\cdot\boldsymbol{r}_j}e^{i(\omega_2-\omega_{q_j})t}a_{q_j}|e_2^j\rangle\langle g_1^j| \\ & + i\Omega_2e^{-i\Delta_2t}\sum_{j=1}^{N}e^{-ik_{22}z_j}|e_2^j\rangle\langle g_2^j| + h.c. \end{aligned} \tag{B.1}$$

The initial state of this process is the final state of the previous one (duration T_2 is assumed for the second process):

$$\begin{aligned} |\phi_3(0)\rangle &= |\phi_2(T_2)\rangle \\ &= x_0^{(3)}(0)|g_1^1\cdots g_1^N\rangle|0\rangle + \cdots + \sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}} Z^{(3)}_{2k_ik_{ii}}(0)|\cdots g_2^i\cdots g_2^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle. \end{aligned} \tag{B.2}$$

We assume that the state vector at time in this process has the most general form:

$$
\begin{aligned}
|\phi_3(t)\rangle = {} & x_0^{(3)}(t)|g_1^1\cdots g_1^N\rangle|0\rangle + \sum_{i=1}^{N} x_i^{(3)}(t)|\cdots e_1^i\cdots\rangle|0\rangle + \sum_{i=1}^{N}\sum_{ii\neq i}^{N} x_{i,ii}^{(3)}(t)|\cdots e_1^i\cdots e_1^{ii}\cdots\rangle|0\rangle \\
& + \sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}} Z_{2k_ik_{ii}}^{(3)}(t)|\cdots g_2^i\cdots g_2^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle + \sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i} Z_{4k_i,ii}^{(3)}(t)|\cdots e_2^i\cdots e_1^{ii}\cdots\rangle|1_{k_i}\rangle \\
& + \sum_{i=1}^{N}\sum_{k_i} y_{2k_i}^{(3)}(t)|\cdots g_2^i\cdots\rangle|1_{k_i}\rangle + \sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i} y_{4k_i,ii}^{(3)}(t)|\cdots g_2^i\cdots e_1^{ii}\cdots\rangle|1_{k_i}\rangle \\
& + \sum_{i=1}^{N}\sum_{k_i} R_{0k_i}^{(3)}(t)|\cdots e_2^i\cdots\rangle|1_{k_i}\rangle \quad + \sum_{i=1}^{N}\sum_{k_i}\sum_{q_i} R_{1k_iq_i}^{(3)}(t)|\cdots g_1^i\cdots\rangle|1_{k_i}\rangle|1_{q_i}\rangle \\
& + \sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}} T_{1k_ik_{ii}}^{(3)}(t)|\cdots e_2^i\cdots g_2^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle + \sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{q_i} T_{3k_iq_i,ii}^{(3)}(t)|\cdots g_1^i\cdots e_1^{ii}\cdots\rangle|1_{k_i}\rangle|1_{q_i}\rangle \\
& + \sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}}\sum_{q_i} T_{6k_ik_{ii}q_i}^{(3)}(t)|\cdots g_1^i\cdots g_2^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle|1_{q_i}\rangle \\
& + \sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}} T_{8k_ik_{ii}}^{(3)}(t)|\cdots e_2^i\cdots e_2^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle + \sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}}\sum_{q_i} T_{10k_ik_{ii}q_i}^{(3)}(t)|\cdots g_1^i\cdots e_2^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle|1_{q_i}\rangle \\
& + \sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}}\sum_{q_i}\sum_{q_{ii}} T_{12k_ik_{ii}q_iq_{ii}}^{(3)}(t)|\cdots g_1^i\cdots g_1^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle|1_{q_i}1_{q_{ii}}\rangle.
\end{aligned} \tag{B.3}
$$

The Schrödinger equation $i\dot{\phi} = V\phi$ gives:

$$\dot{x}_0^{(3)}(t) = 0, \tag{B.4}$$

$$\dot{x}_i^{(3)}(t) = \sum_{k_i} g_{k_i}^{\prime *} e^{i\boldsymbol{k}_i\cdot\boldsymbol{r}_i} e^{i(\omega_1-\omega_{k_i})t} y_{2k_i}^{(3)}(t), \tag{B.5a}$$

$$\dot{y}_{2k_i}^{(3)}(t) = -g'_{k_i} e^{-i\boldsymbol{k}_i\cdot\boldsymbol{r}_i} e^{-i(\omega_1-\omega_{k_i})t} x_i^{(3)}(t) - \Omega_2^* e^{ik_{22}z_i} e^{i\Delta_2 t} R_{0k_i}^{(3)}(t), \tag{B.5b}$$

$$\dot{R}_{0k_i}^{(3)}(t) = \sum_{q_i} g_{q_i}^{\prime *} e^{i\boldsymbol{q}_i\cdot\boldsymbol{r}_i} e^{i(\omega_2-\omega_{q_i})t} R_{1k_iq_i}^{(3)}(t) + \Omega_2 e^{-ik_{22}z_i} e^{-i\Delta_2 t} y_{2k_i}^{(3)}(t), \tag{B.5c}$$

$$\dot{R}_{1k_iq_i}^{(3)}(t) = -g'_{q_i} e^{-i\boldsymbol{q}_i\cdot\boldsymbol{r}_i} e^{-i(\omega_2-\omega_{q_i})t} R_{0k_i}^{(3)}(t), \tag{B.5d}$$

$$\dot{x}_{i,ii}^{(3)}(t) = \sum_{k_i} g_{k_i}^{\prime *} e^{i\boldsymbol{k}_i\cdot\boldsymbol{r}_i} e^{i(\omega_1-\omega_{k_i})t} y_{4k_i,ii}^{(3)}(t), \tag{B.6a}$$

$$\dot{y}_{4k_i,ii}^{(3)}(t) = \sum_{k_{ii}} g_{k_{ii}}^{\prime *} e^{i\boldsymbol{k}_{ii}\cdot\boldsymbol{r}_{ii}} e^{i(\omega_1-\omega_{k_{ii}})t} Z_{2k_ik_{ii}}^{(3)}(t) - 2g'_{k_i} e^{-i\boldsymbol{k}_i\cdot\boldsymbol{r}_i} e^{-i(\omega_1-\omega_{k_i})t} x_{i,ii}^{(3)}(t) - \Omega_2^* e^{ik_{22}z_i} e^{i\Delta_2 t} Z_{4k_i,ii}^{(3)}(t), \tag{B.6b}$$

$$\dot{Z}_{2k_ik_{ii}}^{(3)}(t) = -g'_{k_{ii}} e^{-i\boldsymbol{k}_{ii}\cdot\boldsymbol{r}_{ii}} e^{-i(\omega_1-\omega_{k_{ii}})t} y_{4k_i,ii}^{(3)}(t) - \Omega_2^* e^{ik_{22}z_i} e^{i\Delta_2 t} T_{1k_ik_{ii}}^{(3)}(t), \tag{B.6c}$$

$$\dot{Z}_{4k_i,ii}^{(3)}(t) = \sum_{k_{ii}} g_{k_{ii}}^{\prime *} e^{i\boldsymbol{k}_{ii}\cdot\boldsymbol{r}_{ii}} e^{i(\omega_1-\omega_{k_{ii}})t} T_{1k_ik_{ii}}^{(3)}(t) + \sum_{q_i} g_{q_i}^{\prime *} e^{i\boldsymbol{q}_i\cdot\boldsymbol{r}_i} e^{i(\omega_2-\omega_{q_i})t} T_{3k_iq_i,ii}^{(3)}(t) + \Omega_2 e^{-ik_{22}z_i} e^{-i\Delta_2 t} y_{4k_i,ii}^{(3)}(t), \tag{B.6d}$$

$$
\begin{aligned}
\dot{T}_{1k_ik_{ii}}^{(3)}(t) = {} & \sum_{q_i} g_{q_i}^{\prime *} e^{i\boldsymbol{q}_i\cdot\boldsymbol{r}_i} e^{i(\omega_2-\omega_{q_i})t} T_{6k_ik_{ii}q_i}^{(3)}(t) - g'_{k_{ii}} e^{-i\boldsymbol{k}_{ii}\cdot\boldsymbol{r}_{ii}} e^{-i(\omega_1-\omega_{k_{ii}})t} Z_{4k_i,ii}^{(3)}(t) \\
& + 2\Omega_2 e^{-ik_{22}z_i} e^{-i\Delta_2 t} Z_{2k_ik_{ii}}^{(3)}(t) - 2\Omega_2^* e^{ik_{22}z_i} e^{i\Delta_2 t} T_{8k_ik_{ii}}^{(3)}(t),
\end{aligned} \tag{B.6e}
$$

$$\dot{T}^{(3)}_{3k_iq_i,ii}(t)=\sum_{k_{ii}} g'^{*}_{k_{ii}} e^{i\boldsymbol{k}_{ii}\cdot\boldsymbol{r}_{ii}} e^{i(\omega_1-\omega_{k_{ii}})t} T^{(3)}_{6k_ik_{ii}q_i}(t) - g'_{q_i} e^{-i\boldsymbol{q}_i\cdot\boldsymbol{r}_i} e^{-i(\omega_2-\omega_{q_i})t} Z^{(3)}_{4k_i,ii}(t), \tag{B.6f}$$

$$\dot{T}^{(3)}_{6k_ik_{ii}q_i}(t)=-g'_{q_i} e^{-i\boldsymbol{q}_i\cdot\boldsymbol{r}_i} e^{-i(\omega_2-\omega_{q_i})t} T^{(3)}_{1k_ik_{ii}}(t) - g'_{k_{ii}} e^{-i\boldsymbol{k}_{ii}\cdot\boldsymbol{r}_{ii}} e^{-i(\omega_1-\omega_{k_{ii}})t} T^{(3)}_{3k_iq_i,ii}(t) - \Omega_2^* e^{ik_{22}z_i} e^{i\Delta_2 t} T^{(3)}_{10k_ik_{ii}q_i}(t), \tag{B.6g}$$

$$\dot{T}^{(3)}_{8k_ik_{ii}}(t)=\sum_{q_i} g'^{*}_{q_i} e^{i\boldsymbol{q}_i\cdot\boldsymbol{r}_i} e^{i(\omega_2-\omega_{q_i})t} T^{(3)}_{10k_ik_{ii}q_i}(t) + \Omega_2 e^{-ik_{22}z_i} e^{-i\Delta_2 t} T^{(3)}_{1k_ik_{ii}}(t), \tag{B.6h}$$

$$\dot{T}^{(3)}_{10k_ik_{ii}q_i}(t)=\sum_{q_{ii}} g'^{*}_{q_{ii}} e^{i\boldsymbol{q}_{ii}\cdot\boldsymbol{r}_{ii}} e^{i(\omega_2-\omega_{q_{ii}})t} T^{(3)}_{12k_ik_{ii}q_iq_{ii}}(t) - 2g'_{q_i} e^{-i\boldsymbol{q}_i\cdot\boldsymbol{r}_i} e^{-i(\omega_2-\omega_{q_i})t} T^{(3)}_{8k_ik_{ii}}(t) + \Omega_2 e^{-ik_{22}z_i} e^{-i\Delta_2 t} T^{(3)}_{6k_ik_{ii}q_i}(t), \tag{B.6i}$$

$$\dot{T}^{(3)}_{12k_ik_{ii}q_iq_{ii}}(t)=-g'_{q_{ii}} e^{-i\boldsymbol{q}_{ii}\cdot\boldsymbol{r}_{ii}} e^{-i(\omega_2-\omega_{q_{ii}})t} T^{(3)}_{10k_ik_{ii}q_i}(t). \tag{B.6j}$$

Substituting the equation (B.5d) into (B.5c) and replacing $R^{(3)}_{0k_i}(t)$ by $R'^{(3)}_{0k_i}(t)e^{-i\Delta_2 t}$, we get a new formula:

$$\dot{R}'^{(3)}_{0k_i}(t)+\left(\frac{1}{2}\gamma_2'-i\Delta_2\right)R'^{(3)}_{0k_i}(t)=\Omega_2 e^{-ik_{22}z_0} y^{(3)}_{2k_i}(t). \tag{B.7a}$$

Combining this equation with (B.5b), one can obtain the quadratic equation for$R'^{(3)}_{0k_i}(t)$:

$$\ddot{R}'^{(3)}_{0k_i}(t)+\left(\frac{1}{2}\gamma_2'-i\Delta_2\right)\dot{R}'^{(3)}_{0k_i}(t)+|\Omega_2|^2 R'^{(3)}_{0k_i}(t)+\Omega_2 e^{-ik_{22}z_0} g_{k_i} e^{-i\boldsymbol{k}_i\cdot\boldsymbol{r}_0} e^{-i(\omega_1-\omega_{k_i})t} x^{(3)}_i(t)=0. \tag{B.7b}$$

This equation has the solution like this:

$$R'^{(3)}_{0k_i}(t)=\alpha e^{\lambda_3 t}+\beta e^{\lambda_4 t}+\frac{\Omega_2 e^{-ik_{22}z_0}}{\lambda_3-\lambda_4}\int_0^t g'_{k_i} e^{-i\boldsymbol{k}_i\cdot\boldsymbol{r}_0} e^{-i(\omega_1-\omega_{k_i})t'}[e^{\lambda_4(t-t')}-e^{\lambda_3(t-t')}]x^{(3)}_i(t')dt', \tag{B.7c}$$

and so,

$$y^{(3)}_{2k_i}(t)=\frac{\alpha}{\Omega_2 e^{-ik_{22}z_0}}(\lambda_3+\frac{1}{2}\gamma_2'-i\Delta_2)e^{\lambda_3 t}+\frac{\beta}{\Omega_2 e^{-ik_{22}z_0}}(\lambda_4+\frac{1}{2}\gamma_2'-i\Delta_2)e^{\lambda_4 t} + \frac{1}{\lambda_3-\lambda_4}\int_0^t g'_{k_i} e^{-i\boldsymbol{k}_i\cdot\boldsymbol{r}_0} e^{-i(\omega_1-\omega_{k_i})t'}[(\lambda_4+\frac{1}{2}\gamma_2'-i\Delta_2)e^{\lambda_4(t-t')}-(\lambda_3+\frac{1}{2}\gamma_2'-i\Delta_2)e^{\lambda_3(t-t')}]x^{(3)}_i(t')dt', \tag{B.7d}$$

$$R^{(3)}_{0k_i}(t)=\alpha e^{(\lambda_3-i\Delta_2)t}+\beta e^{(\lambda_4-i\Delta_2)t}+\frac{\Omega_2 e^{-ik_{22}z_0}}{\lambda_3-\lambda_4}e^{-i\Delta_2 t}\int_0^t g'_{k_i} e^{-i\boldsymbol{k}_i\cdot\boldsymbol{r}_0} e^{-i(\omega_1-\omega_{k_i})t'}[e^{\lambda_4(t-t')}-e^{\lambda_3(t-t')}]x^{(3)}_i(t')dt'. \tag{B.7e}$$

λ_3 and λ_4 are the two roots of quadratic equation: $\lambda^2+(\frac{1}{2}\gamma_2'-i\Delta_2)\lambda+|\Omega_2|^2=0$. The two constants α and β are determined by the initial condition (B.2), they are:

$$\alpha=-\beta=\frac{\Omega_2 e^{-ik_{22}z_0}}{\lambda_3-\lambda_4}y^{(3)}_{2k_i}(0). \tag{B.7f}$$

So:

$$y^{(3)}_{2k_i}(t)=y^{(3)}_{2k_i}(0)C(t)-\int_0^t g'_{k_i} e^{-i\boldsymbol{k}_i\cdot\boldsymbol{r}_0} e^{-i(\omega_1-\omega_{k_i})t'} C(t-t')x^{(3)}_i(t')dt', \tag{B.7g}$$

$$R^{(3)}_{0k_i}(t)=y^{(3)}_{2k_i}(0)e^{-\frac{1}{2}\gamma_2' t}D(t)-e^{-\gamma_2' t}\int_0^t g'_{k_i} e^{-i\boldsymbol{k}_i\cdot\boldsymbol{r}_0} e^{-i(\omega_1-\omega_{k_i})t'} D(t-t')x^{(3)}_i(t')dt'. \tag{B.7h}$$

Here we symbolize the two functions $A(t)=\frac{\lambda_3 e^{\lambda_4 t}-\lambda_4 e^{\lambda_3 t}}{\lambda_3-\lambda_4}$ and $\frac{\Omega_2 e^{-ik_{22}z_0}(e^{-\lambda_4 t}-e^{-\lambda_3 t})}{\lambda_3-\lambda_4}$ by $C(t)$ and $D(t)$, respectively. Substituting equation (B.6g) into (B.5a), we can obtain:

$$x^{(3)}_i(t)=x^{(3)}_i(0)e^{-\frac{1}{2}\gamma_1' t}. \tag{B.7i}$$

The expressions for $y^{(3)}_{2k_i}(t)$, $R^{(3)}_{0k_i}(t)$ and $R^{(3)}_{1k_iq_i}(t)$ can be obtained through relations (B.7g), (B.7h) and (B.5d).

Substituting (B.6j) to (B.6i), (B.6i) to (B.6g), (B.6h) to (B.6f) and (B.6e), (B.6e) and (B.6f) to (B.6d), (B.6c) to (B.6b), the differential equations in group (B.6) turn to:

$$\dot{x}^{(3)}_{i,ii}(t) = \sum_{k_i} g'^{*}_{k_i} e^{i\boldsymbol{k}_i\cdot\boldsymbol{r}_0} e^{i(\omega_1-\omega_{k_i})t} y^{(3)}_{4k_i,ii}(t), \tag{B.8a}$$

$$\dot{y}^{(3)}_{4k_i,ii}(t) = -\frac{1}{2}\gamma'_1 y^{(3)}_{4k_i,ii}(t) - 2g'_{k_i} e^{-i\boldsymbol{k}_i\cdot\boldsymbol{r}_0} e^{-i(\omega_1-\omega_{k_i})t} x^{(3)}_{i,ii}(t) - \Omega_2^* e^{ik_{22}z_0} e^{i\Delta_2 t} Z^{(3)}_{4k_i,ii}(t), \tag{B.8b}$$

$$\dot{Z}^{(3)}_{4k_i,ii}(t) = -\frac{1}{2}(\gamma'_1+\gamma'_2) Z^{(3)}_{4k_i,ii}(t) + \Omega_2 e^{-ik_{22}z_0} e^{-i\Delta_2 t} y^{(3)}_{4k_i,ii}(t), \tag{B.8c}$$

$$\dot{Z}^{(3)}_{2k_ik_{ii}}(t) = -g'_{k_{ii}} e^{-i\boldsymbol{k}_{ii}\cdot\boldsymbol{r}_0} e^{-i(\omega_1-\omega_{k_{ii}})t} y^{(3)}_{4k_i,ii}(t) - \Omega_2^* e^{ik_{22}z_0} e^{i\Delta_2 t} T^{(3)}_{1k_ik_{ii}}(t), \tag{B.8d}$$

$$\begin{aligned}\dot{T}^{(3)}_{1k_ik_{ii}}(t) = &-\frac{1}{2}\gamma'_2 T^{(3)}_{1k_ik_{ii}}(t) - g'_{k_{ii}} e^{-i\boldsymbol{k}_{ii}\cdot\boldsymbol{r}_0} e^{-i(\omega_1-\omega_{k_{ii}})t} Z^{(3)}_{4k_i,ii}(t)\\ &+ 2\Omega_2 e^{-ik_{22}z_0} e^{-i\Delta_2 t} Z^{(3)}_{2k_ik_{ii}}(t) - 2\Omega_2^* e^{ik_{22}z_0} e^{i\Delta_2 t} T^{(3)}_{8k_ik_{ii}}(t),\end{aligned} \tag{B.8e}$$

$$\dot{T}^{(3)}_{8k_ik_{ii}}(t) = -\gamma'_2 T^{(3)}_{8k_ik_{ii}}(t) + \Omega_2 e^{-ik_{22}z_0} e^{-i\Delta_2 t} T^{(3)}_{1k_ik_{ii}}(t), \tag{B.8f}$$

$$\dot{T}^{(3)}_{3k_iq_i,ii}(t) = -\frac{1}{2}\gamma'_1 T^{(3)}_{3k_iq_i,ii}(t) - g'_{q_i} e^{-i\boldsymbol{q}_i\cdot\boldsymbol{r}_0} e^{-i(\omega_2-\omega_{q_i})t} Z^{(3)}_{4k_i,ii}(t), \tag{B.8g}$$

$$\begin{aligned}\dot{T}^{(3)}_{6k_ik_{ii}q_i}(t) = &-g'_{q_i} e^{-i\boldsymbol{q}_i\cdot\boldsymbol{r}_0} e^{-i(\omega_2-\omega_{q_i})t} T^{(3)}_{1k_ik_{ii}}(t)\\ &- g'_{k_{ii}} e^{-i\boldsymbol{k}_{ii}\cdot\boldsymbol{r}_0} e^{-i(\omega_1-\omega_{k_{ii}})t} T^{(3)}_{3k_iq_i,ii}(t) - \Omega_2^* e^{ik_{22}z_0} e^{i\Delta_2 t} T^{(3)}_{10k_ik_{ii}q_i}(t),\end{aligned} \tag{B.8h}$$

$$\dot{T}^{(3)}_{10k_ik_{ii}q_i}(t) = -\frac{1}{2}\gamma'_2 T^{(3)}_{10k_ik_{ii}q_i}(t) - 2g'_{q_i} e^{-i\boldsymbol{q}_i\cdot\boldsymbol{r}_0} e^{-i(\omega_2-\omega_{q_i})t} T^{(3)}_{8k_ik_{ii}}(t) + \Omega_2 e^{-ik_{22}z_0} e^{-i\Delta_2 t} T^{(3)}_{6k_ik_{ii}q_i}(t), \tag{B.8i}$$

$$\dot{T}^{(3)}_{12k_ik_{ii}q_iq_{ii}}(t) = -g'_{q_{ii}} e^{-i\boldsymbol{q}_{ii}\cdot\boldsymbol{r}_0} e^{-i(\omega_2-\omega_{q_{ii}})t} T^{(3)}_{10k_ik_{ii}q_i}(t). \tag{B.8j}$$

The three equations (B.8a), (B.8b) and (B.8c) have the solutions as following:

$$x^{(3)}_{i,ii}(t) = x^{(3)}_{i,ii}(0) e^{-\gamma'_1 t}, \tag{B.9a}$$

$$y^{(3)}_{4k_i,ii}(t) = y^{(3)}_{4k_i,ii}(0) C(t) e^{-\frac{1}{2}\gamma'_1 t} - 2x^{(3)}_{i,ii}(0) e^{-\frac{1}{2}\gamma'_1 t} \int_0^t g'_{k_i} e^{-i\boldsymbol{k}_i\cdot\boldsymbol{r}_0} e^{-i(\omega_1-\omega_{k_i})t'} C(t-t') e^{-\frac{1}{2}\gamma'_1 t'} dt', \tag{B.9b}$$

$$Z^{(3)}_{4k_i,ii}(t) = y^{(3)}_{4k_i,ii}(0) D(t) e^{-\frac{1}{2}(\gamma'_1+\gamma'_2)t} - 2x^{(3)}_{i,ii}(0) e^{-\frac{1}{2}(\gamma'_1+\gamma'_2)t} \int_0^t g'_{k_i} e^{-i\boldsymbol{k}_i\cdot\boldsymbol{r}_0} e^{-i(\omega_1-\omega_{k_i})t'} D(t-t') e^{-(\frac{1}{2}\gamma'_1+\lambda_3+\lambda_4)t'} dt'. \tag{B.9c}$$

Replacing $Z^{(3)}_{2k_ik_{ii}}(t)$ and $T^{(3)}_{8k_ik_{ii}}(t)$ by $Z'^{(3)}_{2k_ik_{ii}}(t)e^{i\Delta_2 t}$ and $T'^{(3)}_{8k_ik_{ii}}(t)e^{i\Delta_2 t}$ in equations (B.8d), (B.8e) and (B.8f), we obtain the differential equations:

$$\dot{Z}'^{(3)}_{2k_ik_{ii}}(t) = -e^{-i\Delta_2 t} g'_{k_{ii}} e^{-i\boldsymbol{k}_{ii}\cdot\boldsymbol{r}_0} e^{-i(\omega_1-\omega_{k_{ii}})t} y^{(3)}_{4k_i,ii}(t) - \Omega_2^* e^{ik_{22}z_0} T^{(3)}_{1k_ik_{ii}}(t) - i\Delta_2 Z'^{(3)}_{2k_ik_{ii}}(t), \tag{B.10a}$$

$$\dot{T}^{(3)}_{1k_ik_{ii}}(t) = -\frac{1}{2}\gamma'_2 T^{(3)}_{1k_ik_{ii}}(t) - g'_{k_{ii}} e^{-i\boldsymbol{k}_{ii}\cdot\boldsymbol{r}_0} e^{-i(\omega_1-\omega_{k_{ii}})t} Z^{(3)}_{4k_i,ii}(t) + 2\Omega_2 e^{-ik_{22}z_0} Z'^{(3)}_{2k_ik_{ii}}(t) - 2\Omega_2^* e^{ik_{22}z_0} T'^{(3)}_{8k_ik_{ii}}(t), \tag{B.10b}$$

$$\dot{T}'^{(3)}_{8k_ik_{ii}}(t) = (i\Delta_2 - \gamma'_2) T'^{(3)}_{8k_ik_{ii}}(t) + \Omega_2 e^{-ik_{22}z_0} e^{-i\Delta_2 t} T^{(3)}_{1k_ik_{ii}}(t). \tag{B.10c}$$

The three eigenvalues of the coefficient matrix are $-\frac{1}{2}\gamma_2'$, $-\frac{1}{2}\gamma_2' \pm \sqrt{(\frac{1}{2}\gamma_2' - i\Delta_2)^2 - 4|\Omega_2|^2}$. Based on the initial conditions (B.2), we can write down the time dependent coefficients $Z^{(3)}_{2k_ik_{ii}}(t)$, $T^{(3)}_{1k_ik_{ii}}(t)$ and $T^{(3)}_{8k_ik_{ii}}(t)$ as:

$$\begin{aligned}Z^{(3)}_{2k_ik_{ii}}(t) &= \frac{\Omega_2^*}{\Omega_2} e^{(\lambda_3+\lambda_4)t} \int_0^t g'_{k_i} e^{-i\boldsymbol{k}_i\cdot\boldsymbol{r}_0} e^{-i(\omega_1-\omega_{k_i})t'} e^{\frac{1}{2}\gamma_2' t'} C(t-t')D(t-t') Z^{(3)}_{4k_i,ii}(t')dt' \\ &- \int_0^t g'_{k_i} e^{-i\boldsymbol{k}_i\cdot\boldsymbol{r}_0} e^{-i(\omega_1-\omega_{k_i})t'} (C(t-t'))^2 y^{(3)}_{4k_i,ii}(t')dt' + Z^{(3)}_{2k_ik_{ii}}(0)(C(t))^2,\end{aligned} \tag{B.10d}$$

$$\begin{aligned}T^{(3)}_{1k_ik_{ii}}(t) &= -2e^{-\frac{1}{2}\gamma_2' t} \int_0^t g'_{k_i} e^{-i\boldsymbol{k}_i\cdot\boldsymbol{r}_0} e^{-i(\omega_1-\omega_{k_i})t'} e^{-(\lambda_3+\lambda_4)t'} C(t-t')D(t-t') y^{(3)}_{4k_i,ii}(t')dt' \\ &+ e^{-\frac{1}{2}\gamma_2' t} \int_0^t g'_{k_i} e^{-i\boldsymbol{k}_i\cdot\boldsymbol{r}_0} e^{-i(\omega_1-\omega_{k_i})t'} e^{\frac{1}{2}\gamma_2' t'} \left[2\frac{\Omega_2^*}{\Omega_2} e^{(\lambda_3+\lambda_4)(t-t')}(D(t-t'))^2 - 1\right] Z^{(3)}_{4k_i,ii}(t')dt' \\ &+ 2Z^{(3)}_{2k_ik_{ii}}(0) e^{-\frac{1}{2}\gamma_2' t} C(t)D(t),\end{aligned} \tag{B.10e}$$

$$\begin{aligned}T^{(3)}_{8k_ik_{ii}}(t) &= -e^{-\gamma_2' t} \int_0^t g'_{k_i} e^{-i\boldsymbol{k}_i\cdot\boldsymbol{r}_0} e^{-i(\omega_1-\omega_{k_i})t'} e^{-2(\lambda_3+\lambda_4)t'} (D(t-t'))^2 y^{(3)}_{4k_i,ii}(t')dt' \\ &- e^{-\gamma_2' t} \int_0^t g'_{k_i} e^{-i\boldsymbol{k}_i\cdot\boldsymbol{r}_0} e^{-i(\omega_1-\omega_{k_i})t'} e^{(\frac{1}{2}\gamma_2'-\lambda_3-\lambda_4)t'} D(t-t')C(t'-t) Z^{(3)}_{4k_i,ii}(t')dt' + Z^{(3)}_{2k_ik_{ii}}(0) e^{-\gamma_2' t}(D(t))^2.\end{aligned} \tag{B.10f}$$

The other coefficients are easily obtained through the relations (B.8g)–(B.8j):

$$T^{(3)}_{3k_iq_i,ii}(t) = -e^{-\frac{1}{2}\gamma_1' t} \int_0^t g'_{q_i} e^{-i\boldsymbol{q}_i\cdot\boldsymbol{r}_0} e^{-i(\omega_2-\omega_{q_i})t'} e^{\frac{1}{2}\gamma_1' t'} Z^{(3)}_{4k_i,ii}(t')dt', \tag{B.11a}$$

$$\begin{aligned}T^{(3)}_{6k_ik_{ii}q_i}(t) &= 2\frac{\Omega_2^*}{\Omega_2} e^{(\lambda_3+\lambda_4)t} \int_0^t g'_{q_i} e^{-i\boldsymbol{q}_i\cdot\boldsymbol{r}_0} e^{-i(\omega_2-\omega_{q_i})t'} e^{\frac{1}{2}\gamma_2' t'} D(t-t') T^{(3)}_{8k_ik_{ii}}(t')dt' - \int_0^t g'_{q_i} e^{-i\boldsymbol{q}_i\cdot\boldsymbol{r}_0} \\ &\times e^{-i(\omega_2-\omega_{q_i})t'} C(t-t') T^{(3)}_{1k_ik_{ii}}(t')dt' - \int_0^t g'_{k_{ii}} e^{-i\boldsymbol{k}_{ii}\cdot\boldsymbol{r}_0} e^{-i(\omega_1-\omega_{k_{ii}})t'} C(t-t') T^{(3)}_{3k_iq_i,ii}(t')dt',\end{aligned} \tag{B.11b}$$

$$\begin{aligned}T^{(3)}_{10k_ik_{ii}q_i}(t) &= -e^{-\frac{1}{2}\gamma_2' t} \int_0^t g'_{q_i} e^{-i\boldsymbol{q}_i\cdot\boldsymbol{r}_0} e^{-i(\omega_2-\omega_{q_i})t'} e^{-(\lambda_3+\lambda_4)t'} D(t-t') T^{(3)}_{1k_ik_{ii}}(t')dt' \\ &- e^{-\frac{1}{2}\gamma_2' t} \int_0^t g'_{k_{ii}} e^{-i\boldsymbol{k}_{ii}\cdot\boldsymbol{r}_0} e^{-i(\omega_1-\omega_{k_{ii}})t'} e^{-(\lambda_3+\lambda_4)t'} D(t-t') T^{(3)}_{3k_iq_i,ii}(t')dt' \\ &- 2e^{-\frac{1}{2}\gamma_2' t} \int_0^t g'_{q_i} e^{-i\boldsymbol{q}_i\cdot\boldsymbol{r}_0} e^{-i(\omega_2-\omega_{q_i})t'} e^{\frac{1}{2}\gamma_2' t'} C(t'-t) T^{(3)}_{8k_ik_{ii}}(t')dt',\end{aligned} \tag{B.11c}$$

$$T^{(3)}_{12k_ik_{ii}q_iq_{ii}}(t) = -\int_0^t g'_{q_{ii}} e^{-i\boldsymbol{q}_{ii}\cdot\boldsymbol{r}_0} e^{-i(\omega_2-\omega_{q_{ii}})t'} T^{(3)}_{10k_ik_{ii}q_i}(t')dt'. \tag{B.11d}$$

Appendix C

The time dependent state vector in the third process of the first system is:

$$\begin{aligned}|\phi_3(t)\rangle &= x_0^{(3)}(0)|g_1^1\cdots g_1^N\rangle|0\rangle + \sum_{i=1}^N x_i^{(3)}(0)|\cdots e_1^i\cdots\rangle|0\rangle + \sum_{i=1}^N\sum_{ii\neq i}^N x^{(3)}_{i,ii}(0)|\cdots e_1^i\cdots e_1^{ii}\cdots\rangle|0\rangle \\ &+ \sum_{i=1}^N\sum_{k_i} y^{(3)}_{1k_i}(0)|\cdots g_1^i\cdots\rangle|1_{k_i}\rangle + \sum_{i=1}^N\sum_{ii\neq i}^N\sum_{k_i} y^{(3)}_{3k_i,ii}(0)|\cdots g_1^i\cdots e_1^{ii}\cdots\rangle|1_{k_i}\rangle \\ &+ \sum_{i=1}^N\sum_{k_i} y^{(3)}_{2k_i}(0)C(t)|\cdots g_2^i\cdots\rangle|1_{k_i}\rangle + \sum_{i=1}^N\sum_{k_i} y^{(3)}_{2k_i}(0)D(t)|\cdots e_2^i\cdots\rangle|1_{k_i}\rangle\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i} y^{(3)}_{4k_i,ii}(0)C(t)|\cdots g_2^i\cdots e_1^{ii}\cdots\rangle|1_{k_i}\rangle+\sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i} y^{(3)}_{4k_i,ii}(0)D(t)|\cdots e_2^i\cdots e_1^{ii}\cdots\rangle|1_{k_i}\rangle\\
&+\sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}} Z^{(3)}_{1k_ik_{ii}}(0)|\cdots g_1^i\cdots g_1^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle\\
&+\sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}} Z^{(3)}_{2k_ik_{ii}}(0)(C(t))^2|\cdots g_2^i\cdots g_2^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle\\
&+2\sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}} Z^{(3)}_{2k_ik_{ii}}(0)C(t)D(t)|\cdots g_2^i\cdots e_2^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle\\
&+\sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}} Z^{(3)}_{2k_ik_{ii}}(0)(D(t))^2|\cdots e_2^i\cdots e_2^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle\\
&+\sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}} Z^{(3)}_{3k_ik_{ii}}(0)C(t)|\cdots g_1^i\cdots g_2^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle\\
&+\sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}} Z^{(3)}_{3k_ik_{ii}}(0)D(t)|\cdots g_1^i\cdots e_2^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle. \qquad \text{(C.1)}
\end{aligned}
$$

The steady state of the first system is obtained by taking the limit $t\rightarrow+\infty$ in equation (9),

$$
\begin{aligned}
|\phi(\infty)\rangle=\;&x_0^{(4)}(\infty)|g_1^1\cdots g_1^N\rangle|0\rangle+\sum_{i=1}^{N}\sum_{k_i} y^{(4)}_{1k_i}(\infty)|\cdots g_1^i\cdots\rangle|1_{k_i}\rangle+\sum_{i=1}^{N}\sum_{k_i} y^{(4)}_{2k_i}(\infty)|\cdots g_2^i\cdots\rangle|1_{k_i}\rangle\\
&+\sum_{i=1}^{N}\sum_{k_i}\sum_{q_i} R^{(4)}_{1k_iq_i}(\infty)|\cdots g_1^i\cdots\rangle|1_{k_i}\rangle|1_{q_i}\rangle+\sum_{i=1}^{N}\sum_{k_i}\sum_{q_i} R^{(4)}_{2k_iq_i}(\infty)|\cdots g_2^i\cdots\rangle|1_{k_i}\rangle|1_{q_i}\rangle\\
&+\sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}} Z^{(4)}_{1k_ik_{ii}}(\infty)|\cdots g_1^i\cdots g_1^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle+\sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}} Z^{(4)}_{2k_ik_{ii}}(\infty)|\cdots g_2^i\cdots g_2^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle\\
&+\sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}} Z^{(4)}_{3k_ik_{ii}}(\infty)|\cdots g_1^i\cdots g_2^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle+\sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}}\sum_{q_i} T^{(4)}_{5k_ik_{ii}q_i}(\infty)|\cdots g_1^i\cdots g_1^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle|1_{q_i}\rangle\\
&+\sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}}\sum_{q_i} T^{(4)}_{6k_ik_{ii}q_i}(\infty)|\cdots g_1^i\cdots g_2^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle|1_{q_i}\rangle+\sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}}\sum_{q_i} T^{(4)}_{7k_ik_{ii}q_i}(\infty)\\
&\times|\cdots g_2^i\cdots g_2^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle|1_{q_i}\rangle+\sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}}\sum_{q_i}\sum_{q_{ii}} T^{(4)}_{11k_ik_{ii}q_iq_{ii}}(\infty)|\cdots g_2^i\cdots g_2^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle|1_{q_i}1_{q_{ii}}\rangle\\
&+\sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}}\sum_{q_i}\sum_{q_{ii}} T^{(4)}_{12k_ik_{ii}q_iq_{ii}}(\infty)\times|\cdots g_1^i\cdots g_1^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle|1_{q_i}1_{q_{ii}}\rangle\\
&+\sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}}\sum_{q_i}\sum_{q_{ii}} T^{(4)}_{13k_ik_{ii}q_iq_{ii}}(\infty)|\cdots g_1^i\cdots g_2^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle|1_{q_i}1_{q_{ii}}\rangle. \qquad \text{(C.2)}
\end{aligned}
$$

The time dependent state vector in the last process of the second system is:

$$
\begin{aligned}
|\phi_4(t)\rangle=\;&x_0^{(4)}(0)|g_1^1\cdots g_1^N\rangle|0\rangle+\sum_{i=1}^{N}x_i^{(4)}(0)e^{-\frac{1}{2}\gamma_1' t}|\cdots e_1^i\cdots\rangle|0\rangle+\sum_{i=1}^{N}\sum_{ii\neq i}^{N}x_{i,ii}^{(4)}(0)e^{-\gamma_1' t}|\cdots e_1^i\cdots e_1^{ii}\cdots\rangle|0\rangle\\
&+\sum_{i=1}^{N}\sum_{k_i}[y^{(4)}_{4k_i,ii}(0)-x_i^{(4)}(0)\int_0^t g'_{k_i}e^{-i\boldsymbol{k}_i\cdot\boldsymbol{r}_0}e^{-i(\omega_1-\omega_{k_i})t'}e^{-\frac{1}{2}\gamma_1' t'}dt']|\cdots g_2^i\cdots\rangle|1_{k_i}\rangle\\
&+\sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}e^{-\frac{1}{2}\gamma_1' t}[y^{(4)}_{4k_i,ii}(0)-2x_{i,ii}^{(4)}(0)\int_0^t g'_{k_i}e^{-i\boldsymbol{k}_i\cdot\boldsymbol{r}_0}e^{-i(\omega_1-\omega_{k_i})t'}e^{-\frac{1}{2}\gamma_1' t'}dt']|\cdots g_2^i\cdots e_1^{ii}\cdots\rangle|1_{k_i}\rangle
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}}[2x_{i,ii}^{(4)}(0)\int_0^t g'_{k_{ii}}e^{-i\boldsymbol{k}_{ii}\cdot\boldsymbol{r}_0}e^{-i(\omega_1-\omega_{k_{ii}})t'}e^{-\frac{1}{2}\gamma_1't'}\int_0^{t'}g'_{k_i}e^{-i\boldsymbol{k}_i\cdot\boldsymbol{r}_0}e^{-i(\omega_1-\omega_{k_i})t''}e^{-\frac{1}{2}\gamma_1't''}dt''dt'\\
&+Z_{2k_ik_{ii}}^{(4)}(0)-y_{4k_i,ii}^{(4)}(0)\int_0^t g'_{k_{ii}}e^{-i\boldsymbol{k}_{ii}\cdot\boldsymbol{r}_0}e^{-i(\omega_1-\omega_{k_{ii}})t'}e^{-\frac{1}{2}\gamma_1't'}dt']|\cdots g_2^i\cdots g_2^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle\\
&+\sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}Z_{4k_i,ii}^{(4)}(0)e^{-\frac{1}{2}(\gamma_1'+\gamma_2')t}|\cdots e_2^i\cdots e_1^{ii}\cdots\rangle|1_{k_i}\rangle+\sum_{i=1}^{N}\sum_{k_i}R_{0k_i}^{(4)}(0)e^{-\frac{1}{2}\gamma_2't}|\cdots e_2^i\cdots\rangle|1_{k_i}\rangle\\
&+\sum_{i=1}^{N}\sum_{k_i}\sum_{q_i}[R_{1k_iq_i}^{(4)}(0)-R_{0k_i}^{(4)}(0)\int_0^t g'_{q_i}e^{-i\boldsymbol{q}_i\cdot\boldsymbol{r}_0}e^{-i(\omega_2-\omega_{q_i})t'}e^{-\frac{1}{2}\gamma_2't'}dt']|\cdots g_1^i\cdots\rangle|1_{k_i}\rangle|1_{q_i}\rangle\\
&+\sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}}e^{-\frac{1}{2}\gamma_2't}[T_{1k_ik_{ii}}^{(4)}(0)-Z_{4k_i,ii}^{(4)}(0)\int_0^t g'_{k_{ii}}e^{-i\boldsymbol{k}_i\cdot\boldsymbol{r}_0}e^{-i(\omega_1-\omega_{k_{ii}})t'}\\
&\times e^{-\frac{1}{2}\gamma_1't'}dt']|\cdots g_2^i\cdots e_2^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle-\sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}}\sum_{q_i}[T_{1k_ik_{ii}}^{(4)}(0)\int_0^t g'_{q_i}e^{-i\boldsymbol{q}_i\cdot\boldsymbol{r}_0}e^{-i(\omega_2-\omega_{q_i})t'}e^{-\frac{1}{2}\gamma_2't'}dt'\\
&-Z_{4k_i,ii}^{(4)}(0)\int_0^t g'_{q_i}e^{-i\boldsymbol{q}_i\cdot\boldsymbol{r}_0}e^{-i(\omega_2-\omega_{q_i})t'}e^{-\frac{1}{2}\gamma_2't'}\int_0^{t'}g'_{k_{ii}}e^{-i\boldsymbol{k}_{ii}\cdot\boldsymbol{r}_0}e^{-i(\omega_1-\omega_{k_{ii}})t''}e^{-\frac{1}{2}\gamma_1't''}dt''dt'\\
&-Z_{4k_i,ii}^{(4)}(0)\int_0^t g'_{k_{ii}}e^{-i\boldsymbol{k}_{ii}\cdot\boldsymbol{r}_0}e^{-i(\omega_1-\omega_{k_{ii}})t'}e^{-\frac{1}{2}\gamma_1't'}\int_0^{t'}g'_{q_i}e^{-i\boldsymbol{q}_i\cdot\boldsymbol{r}_0}e^{-i(\omega_2-\omega_{q_i})t''}e^{-\frac{1}{2}\gamma_2't''}dt''dt'\\
&+T_{3k_iq_i}^{(4)}(0)\int_0^t g'_{k_{ii}}e^{-i\boldsymbol{k}_{ii}\cdot\boldsymbol{r}_0}e^{-i(\omega_1-\omega_{k_{ii}})t'}e^{-\frac{1}{2}\gamma_1't'}dt']|\cdots g_2^i\cdots g_1^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle|1_{q_i}\rangle\\
&+\sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{q_i}e^{-\frac{1}{2}\gamma_1't}[T_{3k_iq_i}^{(4)}(0)-Z_{4k_i,ii}^{(4)}(0)\int_0^t g'_{q_i}e^{-i\boldsymbol{q}_i\cdot\boldsymbol{r}_0}e^{-i(\omega_2-\omega_{q_i})t'}e^{-\frac{1}{2}\gamma_2't'}dt']|\cdots e_1^i\cdots g_1^{ii}\cdots\rangle|1_{k_i}\rangle|1_{q_i}\rangle\\
&+\sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}}T_{8k_ik_{ii}}^{(4)}(0)e^{-\gamma_2't}|\cdots e_2^i\cdots e_2^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle+\sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}}\sum_{q_i}e^{-\frac{1}{2}\gamma_2't}[T_{10k_ik_{ii}q_i}^{(4)}(0)\\
&-2T_{8k_ik_{ii}}^{(4)}(0)\int_0^t g'_{q_i}e^{-i\boldsymbol{q}_i\cdot\boldsymbol{r}_0}e^{-i(\omega_2-\omega_{q_i})t'}e^{-\frac{1}{2}\gamma_2't'}dt']|\cdots e_2^i\cdots g_1^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle|1_{q_i}\rangle\\
&+\sum_{i=1}^{N}\sum_{ii\neq i}^{N}\sum_{k_i}\sum_{k_{ii}}\sum_{q_i}\sum_{q_{ii}}[2T_{8k_ik_{ii}}^{(4)}(0)\int_0^t g'_{q_{ii}}e^{-i\boldsymbol{q}_{ii}\cdot\boldsymbol{r}_0}e^{-i(\omega_2-\omega_{q_{ii}})t'}e^{-\frac{1}{2}\gamma_2't'}\int_0^{t'}g'_{q_{ii}}e^{-i\boldsymbol{q}_{ii}\cdot\boldsymbol{r}_0}e^{-i(\omega_2-\omega_{q_{ii}})t''}e^{-\frac{1}{2}\gamma_2't''}dt''dt'\\
&+T_{12k_ik_{ii}q_iq_{ii}}^{(4)}(0)-T_{10k_ik_{ii}q_i}^{(4)}(0)\int_0^t g'_{q_{ii}}e^{-i\boldsymbol{q}_{ii}\cdot\boldsymbol{r}_0}e^{-i(\omega_2-\omega_{q_{ii}})t'}e^{-\frac{1}{2}\gamma_2't'}dt']|\cdots g_1^i\cdots g_1^{ii}\cdots\rangle|1_{k_i}1_{k_{ii}}\rangle|1_{q_i}1_{q_{ii}}\rangle. \qquad \text{(C.3)}
\end{aligned}
$$